Mathematics

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Contents

Chapter 1

Sets of numbers

1.1 Introduction

In this chapter, we'll be looking at *sets of numbers*. Indeed, not all numbers verify the same properties. There are several kinds of numbers that we group together in *sets*. The goal of this chapter will be to present the four main sets and to define the operations within them.

1.2 Natural numbers

1.2.1 Definition

Natural numbers are historically the first ones mankind has needed. They are the positive integers, which form the set denoted by $\mathbb{N}$.

$$\boxed{\mathbb{N} = \{0; 1; 2; 3; 4; ...\}.}$$

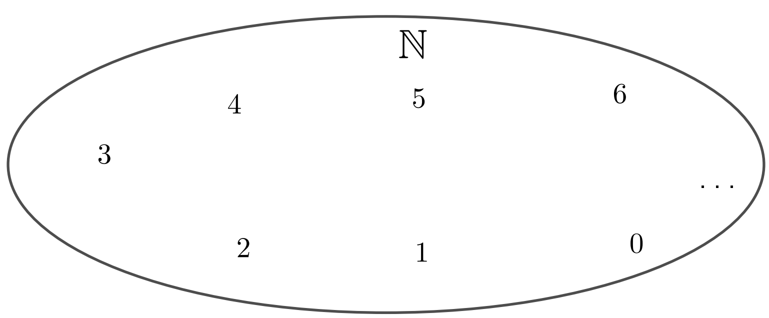


Figure 1.1: Natural numbers set.

Remark. We write $\mathbb{N}^*$ the set of natural numbers without 0.

1.2.2 Order of operations

Let's try to compute $3+5\cdot 4$. Does that make 32 or 23? Actually, it's 23. As a matter of fact, an arithmetic expression is not read from left to right like a english sentence. The different operations are performed in the following order:

1. Parentheses (also named brackets). We start with the nested parentheses (the ones which are inside other brackets).
2. Exponents
3. Multiplication and Division
4. Additions and Subtractions

Memory aid: PEMDAS → Please Excuse My Dear Aunt Sally !

Remark. Operations of same level are executed from left to right.

Example. To compute $2+3\cdot[4+5\cdot(6-2)-7]$, we proceed the following way.

$$\begin{aligned} & 2+3\cdot[4+5\cdot\underbrace{(6-2)}-7] \\ = \; & 2+3\cdot[4+\underbrace{5\cdot\quad(4)}-7] \\ = \; & 2+3\cdot[\underbrace{4+\quad 20-7}] \\ = \; & 2+\underbrace{3\cdot\qquad[17]} \\ = \; & 2+\qquad 51 \\ = \; & 53. \end{aligned}$$

Division by 0

It is not possible to divide a number different from 0 by 0.

Let's see in detail what happens when we try to divide 5 by 0.
Let's assume that we can divide 5 by 0 and the result is equal to x.
This assumption is therefore written:

$$\frac{5}{0}=x.$$

In a corresponding manner, the last equality can also be written as follows

$$5=0\cdot x=0.$$

So, if dividing a nonzero number by 0 were possible, this would imply that $5=0$, which is *impossible*.

Regarding the division of 0 by 0, it leads to an *undetermined form*.
Indeed, let's assume that the quotient of 0 by 0 exists and gives x.
In this situation, we would have

$$\frac{0}{0}=x.$$

This equality could be written in the form

$$0 \cdot x = 0.$$

Consequently, it's impossible to answer the following question:
«A number is multiplied by 0; the result gives 0. What is the number?».

Indeed, any (real) number is an answer to this question. Then, we say that $\frac{0}{0}$ is *undetermined*, because dividing 0 by 0 "could be any number".

In a nutshell,

$5 \cdot 0 = 0,$	$\frac{0}{5} = 0,$	$\frac{5}{0}$ is impossible,	$\frac{0}{0}$ is undetermined.

Exercise 1.1. Give

a) The sum of 12 and 25.
b) The difference between 108 and 73.
c) The product of 7 and 15.
d) The quotient of 84 by 7.
e) Two numbers such that the product gives 42 (give three answers).

Exercise 1.2. Compute without calculator.

a) $4 + 12 : 4 + 12$
b) $21 + 24 : 3 - 3 \cdot 3$
c) $60 + (7 - 6) - 4 \cdot 5$
d) $3 \cdot 4 + 5 \cdot 12 - 6 \cdot 2$
e) $150 - (100 + 50) : 5$
f) $(5 \cdot 3 - 15) : [6 - 3 \cdot 2]$
g) $1 + 19(7 - 8 : 4)$
h) $[5 + 3 \cdot 4] : (16 - 16)$
i) $(3 \cdot 4 + 5) \cdot 12 - 6 \cdot 2$
j) $\{100 - [50 - (40 - 9)]\} \cdot 2$

Exercise 1.3. Add the necessary parentheses.

a) $5 \cdot 18 + 4 = 110$
b) $5 + 3 \cdot 1 + 1 = 16$
c) $80 + 40 : 2 = 100$
d) $2 + 2 \cdot 2 + 2 \cdot 2 = 24$
e) $9 - 9 \cdot 9 + 9 = 9$
f) $3 \cdot 2 + 5 \cdot 6 = 66$
g) $30 : 2 + 8 = 3$
h) $5 - 2 \cdot 9 - 7 = 20$
i) $100 - 1 \cdot 100 - 1 = 9899$
j) $3 \cdot 3 + 3 \cdot 3 + 3 - 3 = 36$

1.3 Integers

1.3.1 Definition

For obvious reasons, the natural numbers set $\mathbb{N}$ is not sufficient to represent all the situations encountered in everyday life (for example temperatures expressed in degrees Celsius, the floor number of something located in the basement of a building, ...). On the other hand, some operations, such as subtraction, are not always defined in $\mathbb{N}$. For example, $2 - 5 = -3 \notin \mathbb{N}$. It is therefore necessary to add to $\mathbb{N}$ the *negative integers*. To specify that a given quantity is less than 0, we add the sign "$-$". The set of all integers (positive and negative) is denoted by $\mathbb{Z}$ and is called *set of integers*.

$$\boxed{\mathbb{Z} = \{...; -4; -3; -2; -1; 0; 1; 2; 3; 4; ...\}.}$$

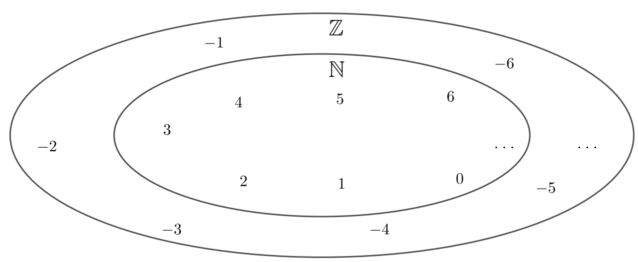


Figure 1.2: Set of integers.

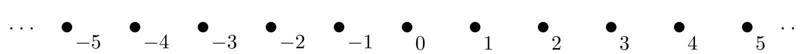


Figure 1.3: Integers.

1.3.2 Sign rules

The product (result of the multiplication) of two integers is done using the *sign rules*:

$$+ \cdot + = +$$
$$+ \cdot - = -$$
$$- \cdot + = -$$
$$- \cdot - = +$$

Remark. Sign rules can be worded using the mnemonic below:

1. The friend $(+)$ of my friend $(+)$ is my friend $(+)$.
2. The friend $(+)$ of my enemy $(-)$ is my enemy. $(-)$.
3. The enemy $(-)$ of my friend $(+)$ is my enemy $(-)$.
4. The enemy $(-)$ of my enemy $(-)$ is my friend $(+)$.

Example.

1. $5 \cdot 3 = 15$.
2. $5 \cdot (-3) = -15$.
3. $(-5) \cdot 3 = -15$.
4. $(-5) \cdot (-3) = 15$.

Remark. The sign of the division of two integers is also based on the sign rules.

Example.

1. $15 : 3 = 5$.
2. $15 : (-3) = -5$.
3. $(-15) : 3 = -5$.
4. $(-15) : (-3) = 5$.

Exercise 1.4. Compute.

a) $(+5) \cdot (-7)$	b) $(+12) : (+4)$
c) $(-25) : (+5)$	d) $(-30) : (-10)$
e) $(-5) \cdot (-8) \cdot (+6)$	f) $(+3) \cdot (-7) \cdot (+11)$
g) $(-8) \cdot (-18) : (+3)$	h) $(-15) : (-5) \cdot (+3)$

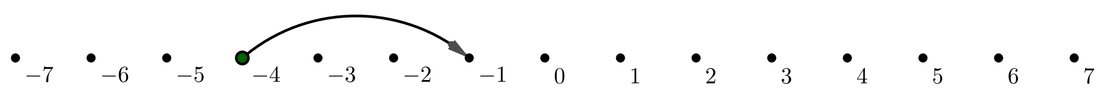
1.3.3 Sum and difference of two integers

The sum of two integers is made by considering the "numerical line".

Example.

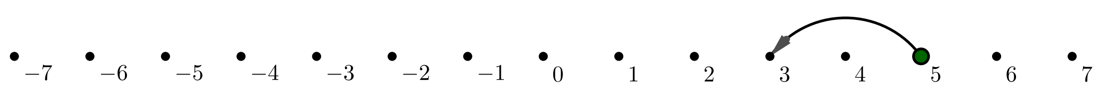
1. $(-4) + 3 = -1$

$(-4) + 3$ involves moving 3 units to the right from -4



2. $5 + (-2) = 5 - 2 = 3$

$5 + (-2)$ involves moving 2 units to the left from 5



Example.

1. $5 - 2 = 3$.
2. $(-5) - 2 = -7$.
3. $5 - (-2) = 5 + 2 = 7$.
4. $(-5) - (-2) = -5 + 2 = -3$.

Definition. Two integers are said to be *opposite* if their sum is zero.

Example.

1. -9 is the opposite of 9.
2. 11 is the opposite -11.

Remark. Geometrically, the opposite of an integer is obtained by central symmetry of centre 0.

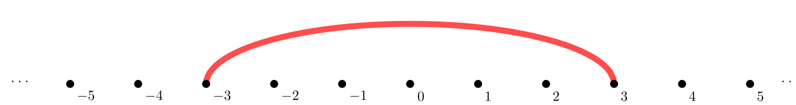


Figure 1.4: Opposite of an integer.

Exercise 1.5. Compute.

a) $(-2)+(+15)$ b) $(+5)+(-4)$
c) $(+8)-(-8)$ d) $(-15)-(+25)-(-5)$
e) $(-7)-(+8)-(-4)$ f) $(-25)-(+36)-(+85)-(-100)$

Exercise 1.6. Compute without calculator.

a) $(-3)-5\cdot(-2)$ b) $(-8)-(-18):(+3)$
c) $25-10:5+5-1\cdot5\cdot12-6\cdot2$ d) $-5\cdot(-2)-(-3):(-1)-(-2)$
e) $-5(-3\cdot7+3)$ f) $[-3-(-2)]:[-13-(-13)]$
g) $0:[-5-(-5)]$ h) $[-5-(-5)]:0$
i) $[(-3-3)-3](-3-3-3)$ j) $(-6+3\cdot2):[4+2\cdot(-2)]$
k) $-1-[(-2):(-1)+2]$ l) $\{[-1-(-2)]:(-1)\}\cdot(-2)$
m) $(+3)\cdot[(-7)+(+11)]:[(-2)-(-8)]$ n) $[-6-(-3)\cdot(-4)]:\{[-7-8:(-2)]\cdot(-6)\}$

Exercise 1.7. Rewrite those expressions with $a=-1$ and compute.

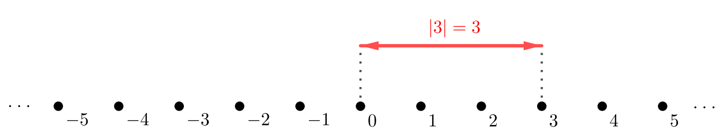
a) $2a+3$ b) $(2a-1)+3$
c) $2a-1+3$ d) $2(a-1)+3$
e) $2a-(1+3)$ f) $2[a-(a+3)]$

1.3.4 Absolute value

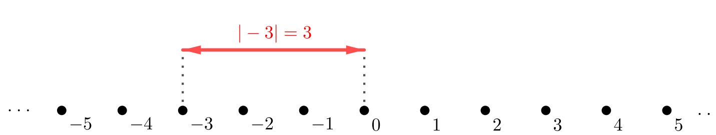
Definition. We call *absolute value* of a number $a\in\mathbb{Z}$ the distance between a and 0. We write $|a|$.

Example.

1. $|3|=3$.



2. $|-3|=3$.



Based on the above examples, the following theorem is deduced.

Theorem. *If $a \in \mathbb{Z}$, then*

$$\boxed{|a| = \begin{cases} a & \textit{if} \quad a > 0 \\ -a & \textit{if} \quad a < 0 \end{cases}.}$$

In other words, the absolute value of a relative integer is nothing more than the number from which the $-$ sign is removed.

Exercise 1.8. Rewrite the number by clearing the absolute value and simplify the result.

a) $|-11+1|$ b) $|-3-2|$

c) $|7|+|-4|$ d) $|-7|+|4|$

e) $|8|+|-9|$ f) $|-1|+|-9|$

g) $|6|-|-3|$ h) $|-5|-|2|$

i) $\dfrac{6}{|-2|}$ j) $\dfrac{|-6|}{-2}$

k) $4 \cdot |6-7|$ l) $(-5) \cdot |3-6|$

1.4 Rational numbers

1.4.1 Definition

Definition. The division shows three numbers:

— The number which is divided is called the *dividend*.

— The number that divides is called the *divisor*.

— The result of the division is called the *quotient*.

Example. When we divide 18 by 6, we obtain 3.

— The dividend is 18.

— The divisor is 6.

— The quotient is 3.

Exercise 1.9. Give

a) The quotient if the dividend is 12 and the divisor 0.

b) The quotient if the dividend is 0 and the divisor 12.

c) The dividend if the divisor is 8 and the quotient 7.

d) The divisor if the dividend is 72 and the quotient 12.

From the moment you want to compare two quantities, you need to make *ratios*, and that means *divisions*. However, it soon becomes clear that a division of two integers doesn't always result in an integer. So $\mathbb{Z}$ doesn't contain all the numbers. For example, the quotient of 3 by 2 gives $1,5$. The quotient of 1 by 3 gives $0,\overline{3} = 0,3333....$ We will admit that any decimal number, with a finite number of decimals or whose decimal development is periodic, can be written as a quotient of two integers. A new set which groups all the results of these divisions is therefore necessary. It is the set of *rational numbers*, denoted by $\mathbb{Q}$.

$$\boxed{\mathbb{Q} = \left\{\frac{a}{b} : a \in \mathbb{Z} \text{ and } b \in \mathbb{Z}^*\right\} = \{\text{Numbers that can be written as a ratio of two integers}\}.}$$

Example.

1. $1,3$ is equal to the quotient of 13 by 10.
2. $0,\overline{45}$ equals the quotient of 5 by 11.
3. -9 is equal to the quotient of -9 by 1.

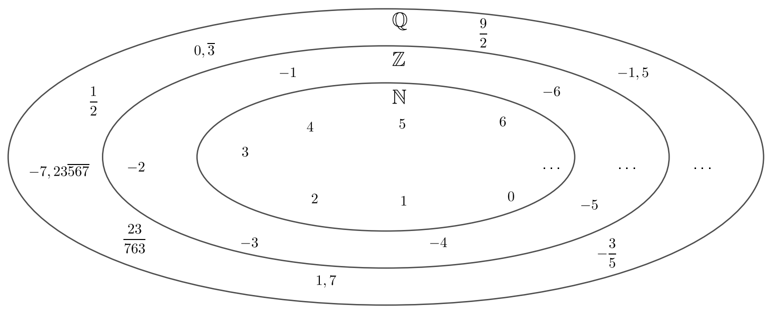


Figure 1.5: Set of the rational numbers.

1.4.2 Fractions

Definition. We call *fraction* any ratio of two integers a and b and we write $\frac{a}{b}$. The number a (located at the top) is called *numerator*, while b (located at the bottom) is the *denominator*.

Example.

1. The number $\frac{3}{4}$ represents the quotient of 3 by 4 and can also be written $0,75$.
2. The number 7 can be also written as $\frac{7}{1}$.
3. The fraction $\frac{1}{3}$ is equal to the number $0,\overline{3}$.
4. The fraction $\frac{1}{7}$ equals $0,\overline{142857}$.

Remark. When we divide 1 by 7 using a calculator, the result is "0.142857143". It is therefore not easy to guess that this is actually the periodic decimal number $0,\overline{142857}$. It is consequently wrong to write

$$\frac{1}{7} = 0.142857143.$$

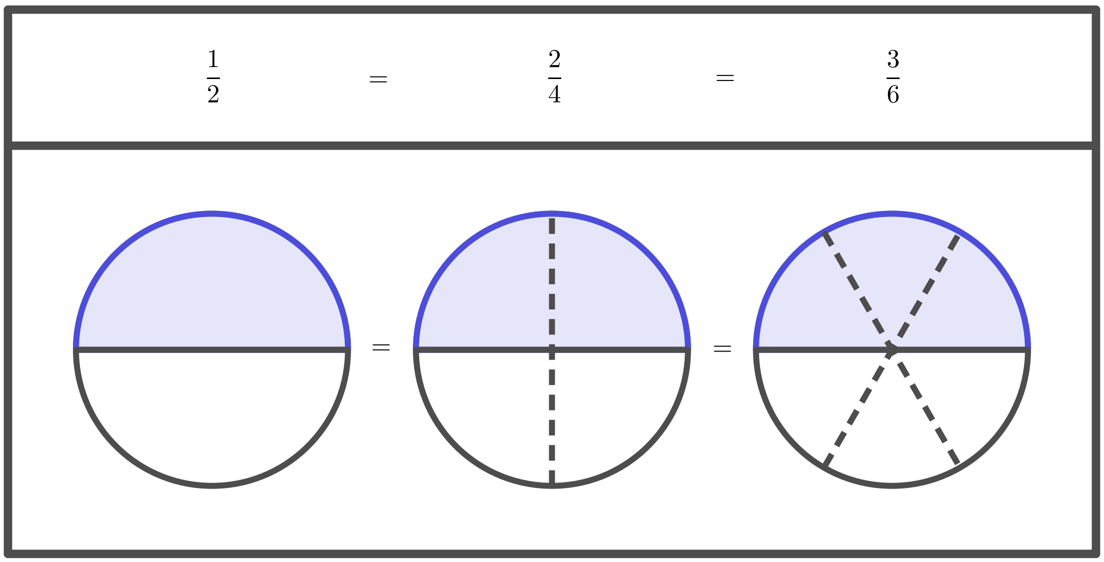
but correct to write

$$\frac{1}{7} \cong 0.142857143.$$

One would therefore prefer to write a rational number as a fraction rather than as a decimal number, since such numbers frequently lead to approximations, as shown in the above example.

1.4.3 Expansion and simplification

The fraction $\frac{1}{2}$ represents the decimal number $0,5$. A real-life situation that could be symbolized by this same fraction is the use of one out of two slices of a cake. If this cake had been cut into 4 identical slices, 2 would have been needed to ensure that the amount of cake eaten remained the same. Likewise, it is the same as taking 3 out of 6 identical slices.



In other words, if the number of slices available doubles, then so does the number of slices to be eaten. More generally, if we multiply by x (for example, we triple, quadruple, multiply by 37, etc.) the total number of slices available, then we will eat x times more slices than in the initial situation.

Mathematically speaking, this observation means that the fractions $\frac{3}{4}$, $\frac{6}{8}$, $\frac{9}{12}$, or $\frac{300}{400}$, are equal. In fact, each of these quotients gives the decimal number $0,75$.

Definition.

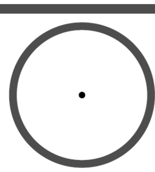
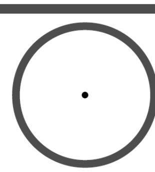
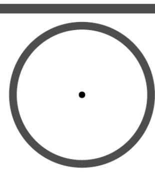
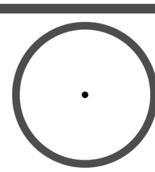
1. *Expanding* a fraction corresponds to multiplying its numerator and denominator by the same integer.
2. *Simplifying* a fraction means dividing its numerator and denominator by the same integer.
3. A fraction is said *irreducible* when it's no longer possible to simplify it.

Remark. When we expand or simplify a fraction, we obtain a new fraction which is equal to the original one.

Example.

1. Let's expand $\frac{2}{7}$ by 3:
$$\frac{2}{7} = \frac{2 \cdot 3}{7 \cdot 3} = \frac{6}{21}.$$
2. Let's simplify $\frac{25}{35}$ by 5:
$$\frac{25}{35} = \frac{25 : 5}{35 : 5} = \frac{5}{7}.$$
3. Let's simplify $\frac{11}{22}$ as much as possible:
$$\frac{11}{22} \overset{\text{by } 11}{=} \frac{1}{2}.$$
4. Let's simplify $\frac{560}{700}$ as much as possible:
$$\frac{560}{700} \overset{\text{by } 10}{=} \frac{56}{70} \overset{\text{by } 7}{=} \frac{8}{10} \overset{\text{by } 2}{=} \frac{4}{5}.$$

Exercise 1.10. Answer the following questions. Give a geometric justification and an algebraic justification.

Alice eats 1/2 pizza	Alain eats 2/5 pizza	Who eats the most?
		
Alice eats 3/4 pizza	Alain eats 4/5 pizza	Who eats the most?
		

Exercise 1.11. Expand the fractions below as indicated.

a) $\frac{7}{5} = \frac{}{25}$ b) $\frac{16}{18} = \frac{64}{}$

c) $\frac{8}{20} = \frac{}{100}$ d) $\frac{3}{14} = \frac{}{42}$

e) $\frac{24}{11} = \frac{}{121}$ f) $\frac{4}{8} = \frac{}{10}$

g) $\frac{30}{24} = \frac{}{32}$ h) $\frac{27}{63} = \frac{}{77}$

i) $\frac{27}{36} = \frac{}{28}$ j) $\frac{21}{15} = \frac{}{25}$

Exercise 1.12. Simplify the fractions as much as possible.

a) $\frac{15}{20}$	b) $\frac{14}{6}$
c) $\frac{27}{21}$	d) $\frac{12}{16}$
e) $\frac{22}{28}$	f) $\frac{25}{15}$
g) $\frac{42}{39}$	h) $\frac{15}{25}$
i) $\frac{40}{45}$	j) $\frac{18}{24}$
k) $\frac{35}{56}$	l) $\frac{13}{169}$
m) $\frac{36}{90}$	n) $\frac{40}{384}$
o) $\frac{96}{900}$	p) $\frac{640}{48}$
q) $\frac{1210}{330}$	r) $\frac{96}{360}$
s) $\frac{768}{64}$	t) $\frac{608}{432}$

Exercise 1.13.

1. Give five fractions equivalent to $\frac{3}{4}$ and also five fractions equivalent to $\frac{5}{6}$.
2. For each of these fractions, give an equivalent fraction whose
   a) denominator is 120.
   b) numerator is 120.
   c) denominator is a power of 12.
   d) denominator is a power of 10.
   e) numerator is the same and is between 100 and 110.
   f) denominators are the same and are between 50 and 60.

Exercise 1.14. For each cases, determine which of the two fractions is greater, after putting both fractions in the same denominator.

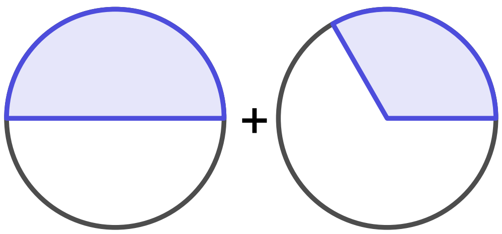
a) $\frac{5}{8}$ and $\frac{6}{19}$	b) $\frac{7}{15}$ and $\frac{5}{12}$
c) $\frac{9}{20}$ and $\frac{11}{18}$	d) $\frac{11}{36}$ and $\frac{9}{32}$

1.4.4 Addition and subtraction

The sum and difference of two fractions can be illustrated by the example below.

Example.

Let's compute $\dfrac{1}{2}+\dfrac{1}{3}$.



These two "slices" are not of the same size, so it is not convenient to add them together. By comparison, one might ask what 1 franc + 2 euros is worth. To overcome this problem, it is necessary to expand the fractions to put them at the *same denominator*, as shown in the figure below.

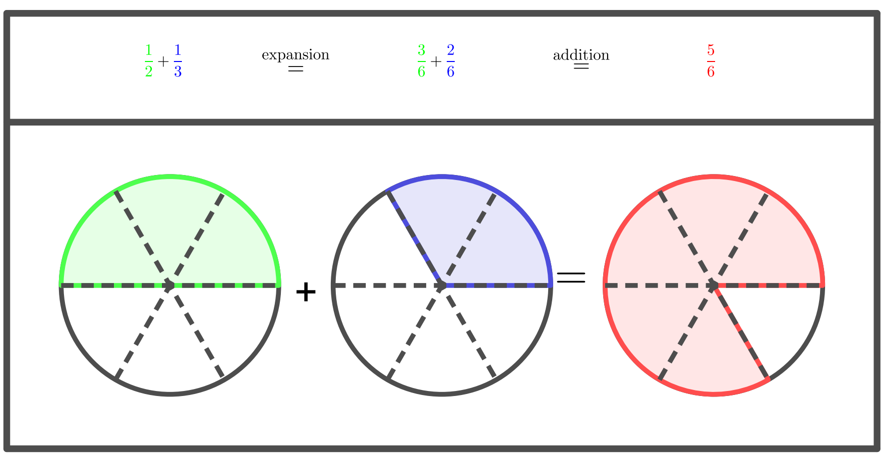


Figure 1.6: Sum of two fractions.

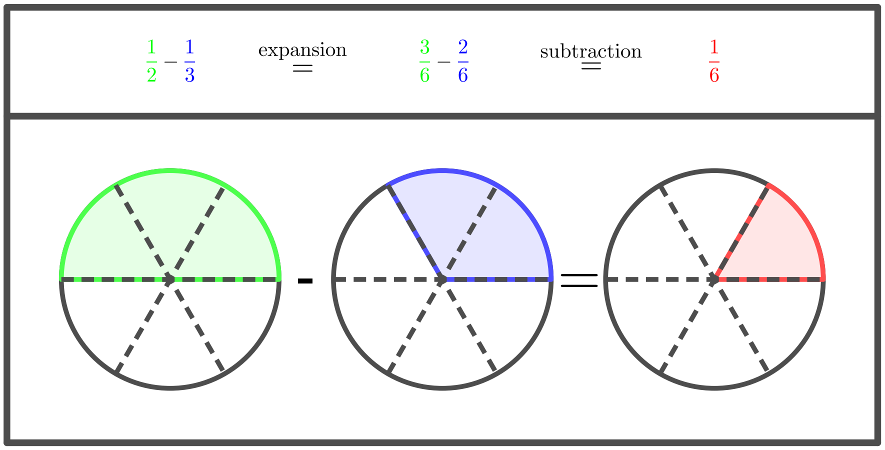


Figure 1.7: Difference of two fractions.

General case: As shown in the examples above, in order to add two fractions, we first have to put them in the **same denominator** (which will be a multiple of each of the denominators of the initial fractions), then add only the numerators and simplify the resulting fraction as much as possible if necessary.

By convention, the result will always be written as an irreducible fraction or an integer.

Example.

1. $\frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}=\frac{8+9}{12}=\frac{17}{12}$.
2. $\frac{1}{15}+\frac{1}{10}-\frac{5}{6}=\frac{2}{30}+\frac{3}{30}-\frac{25}{30}=\frac{2+3-25}{30}=\frac{-20}{30}=\frac{-2}{3}=-\frac{2}{3}$.

Remark. The fraction $\frac{-1}{2}$ represents the decimal number $-0,5$. The same applies to the fraction $\frac{1}{-2}$. Since these two fractions are identical, it is the custom to write it as $-\frac{1}{2}$. In other words, we have

$$-0,5=\frac{-1}{2}=\frac{1}{-2}=-\frac{1}{2}.$$

Remark. Let's compute

$$\frac{5}{36}+\frac{7}{54}.$$

To put the two fractions in the same denominator, two options are possible:

1. **Multiply 36 by 54 in order to get a common denominator**
   We have
   $$\frac{5}{36}+\frac{7}{54}=\frac{270}{1944}+\frac{252}{1944}=\frac{522}{1944}\overset{\text{par } 18}{=}\frac{29}{108}.$$
2. **Determine the least common multiple of 36 and 54**
   We have
   $$\frac{5}{36}+\frac{7}{54}=\frac{15}{108}+\frac{14}{108}=\frac{29}{108}.$$

Exercise 1.15. Compute the expressions below and simplify if necessary.

a) $\frac{1}{3}+\frac{2}{3}$

b) $\frac{5}{6}+\frac{3}{6}-\frac{7}{6}$

c) $\frac{4}{3}+\frac{3}{2}$

d) $\frac{4}{20}+\frac{27}{15}$

e) $\frac{2}{21}+\frac{7}{4}$

f) $\frac{3}{25}+\frac{25}{3}$

g) $\frac{12}{42}+\frac{15}{6}$

h) $\frac{9}{10}-\frac{8}{45}$

i) $\frac{11}{4}+\frac{-2}{3}$

j) $\frac{5}{6}+\frac{12}{-15}$

k) $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

l) $\frac{1}{4}+\frac{3}{16}+\frac{5}{32}$

m) $\frac{5}{6}+\left(-\frac{4}{5}\right)-\left(-\frac{2}{15}\right)$

n) $\frac{5}{12}+\frac{7}{18}+\frac{8}{3}+\frac{13}{36}$

o) $\frac{7}{5}+\frac{3}{4}+\frac{13}{20}+1$

p) $\frac{9}{20}+\frac{37}{50}+\frac{63}{10}+\frac{3}{25}$

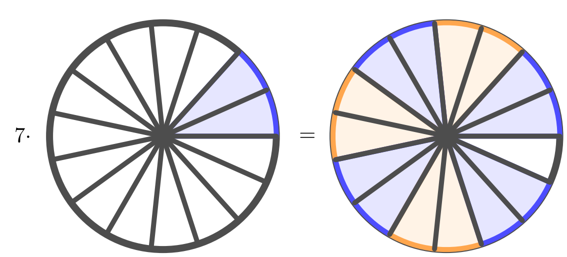
1.4.5 Multiplication and division

Product of two fractions

The product of two fractions can be illustrated by the two examples below.

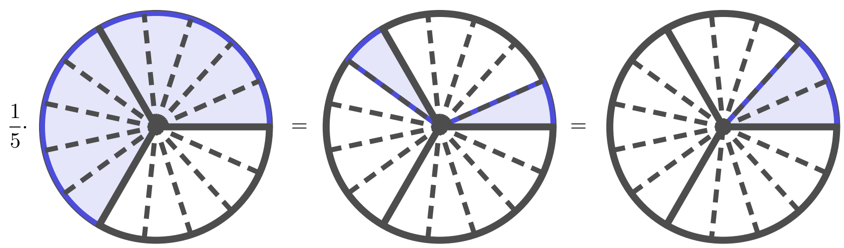
Example.

$$7\cdot\frac{2}{15}=\frac{7\cdot 2}{15}=\frac{14}{15}$$



Example.

$$\frac{1}{5}\cdot\frac{2}{3}=\frac{2}{5\cdot 3}=\frac{2}{15}$$



The above examples show that the product of two fractions is obtained by multiplying the numerators by each other and doing the same with the denominators.

In other words, we have

$$\boxed{\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}.}$$

Remark. It is advised to simplify as much as possible before computing the product.

Example.

1. $\frac{3}{7}\cdot\frac{5}{4}=\frac{3\cdot 5}{7\cdot 4}=\frac{15}{28}.$
2. $\frac{{}^{1}\cancel{10}}{{}^{3}\cancel{21}}\cdot\frac{{}^{1}\cancel{7}}{{}^{5}\cancel{50}}=\frac{1\cdot 1}{3\cdot 5}=\frac{1}{15}.$

Exercise 1.16. Compute and simplify if needed.

a) $\frac{3}{4}\cdot\frac{7}{3}$

b) $\frac{1}{8}\cdot\frac{6}{16}$

c) $\frac{35}{21}\cdot\frac{7}{20}$

d) $\frac{21}{54}\cdot\frac{27}{28}$

e) $\frac{40}{42}\cdot\frac{21}{56}$

f) $\frac{9}{3}\cdot 25$

g) $18\cdot\frac{4}{72}$

h) $\left(-\frac{3}{5}\right)\cdot\left(-\frac{7}{12}\right)$

i) $-5\cdot\frac{4}{7}$

j) $\left(-\frac{7}{9}\right)\cdot 2$

k) $\left(-\frac{1}{3}\right)\cdot\left(-\frac{2}{9}\right)\cdot(-2)$

l) $\frac{1}{7}\cdot\frac{14}{9}\cdot\frac{7}{2}\cdot\frac{9}{4}$

m) $\frac{14}{19}\cdot\frac{19}{4}\cdot\frac{5}{4}\cdot\frac{4}{5}\cdot 0$

n) $\frac{3}{4}\cdot 13\cdot\frac{12}{9}\cdot\frac{41}{41}\cdot\frac{1}{39}$

Quotient of two fractions

Theorem. *Dividing a fraction by* $\frac{b}{a}$ *is like multiplying it by its reciprocal* $\frac{a}{b}$.

Example.

1. $2 : \frac{5}{11} = 2 \cdot \frac{11}{5} = \frac{22}{5}$.
2. $\frac{4}{3} : 2 = \frac{{}^{2}\cancel{4}}{3} \cdot \frac{1}{{}^{1}\cancel{2}} = \frac{2}{3}$.
3. $5 : \frac{3}{2} = \frac{5}{1} \cdot \frac{2}{3} = \frac{10}{3}$.
4. $\frac{1}{5} : \frac{2}{7} : \frac{8}{5} = \frac{1}{{}^{1}\cancel{5}} \cdot \frac{7}{2} \cdot \frac{{}^{1}\cancel{5}}{8} = \frac{7}{16}$.
5. $\frac{1}{3} - \frac{2}{3} : \frac{4}{15} + \frac{1}{15} = \frac{1}{3} - \left(\frac{{}^{1}\cancel{2}}{{}^{1}\cancel{3}} \cdot \frac{{}^{5}\cancel{15}}{{}^{2}\cancel{4}}\right) + \frac{1}{15} = \frac{1}{3} - \frac{5}{2} + \frac{1}{15} = \frac{10}{30} - \frac{75}{30} + \frac{2}{30} = \frac{-63}{30} = -\frac{21}{10}$.

Exercise 1.17. Compute.

a) $\frac{4}{5} : \frac{2}{10}$

b) $\frac{4}{7} : \frac{3}{11}$

c) $\frac{2}{5} : \frac{8}{25}$

d) $\frac{\frac{7}{3}}{\frac{5}{9}}$

e) $\frac{1}{2} : 2$

f) $18 : \frac{6}{17}$

g) $\frac{\frac{24}{5}}{64}$

h) $\frac{24}{\frac{5}{4}}$

i) $\frac{25}{60} : \frac{24}{800} : \frac{75}{54}$

j) $\frac{20}{3} : \left(\frac{7}{4} : \frac{14}{3}\right)$

k) $\frac{4}{-9} : \left(\frac{2}{-3} : \frac{-40}{-36}\right)$

l) $-\frac{65}{-121} : \frac{-150}{48} : \frac{-13}{50} \cdot \frac{3}{2}$

Exercise 1.18. Compute and simplify if needed.

a) $\frac{22}{13} \cdot \frac{32}{11} : \frac{17}{26}$

b) $\frac{7}{12} \cdot \frac{5}{14} + \frac{4}{3} \cdot \frac{2}{7}$

c) $\frac{7}{4} \cdot 2 + \frac{8}{15} : 4$

d) $\left(\frac{11}{5} - \frac{3}{20}\right) - \left(\frac{9}{10} - \frac{11}{15}\right)$

e) $\dfrac{\frac{6}{2} + \frac{4}{3}}{3 - \frac{8}{3}}$

f) $\left(\frac{12}{13} : 5\right)\left(\frac{2}{3} - 4\right)$

g) $\frac{1}{2} - \left\{\frac{1}{3} - \left[\frac{2}{3} - \left(\frac{1}{6} - \frac{1}{2}\right)\right]\right\}$

h) $12 - 2\left(-\frac{3}{8} + \frac{4}{5}\right) \cdot 4$

1.4.6 Applications

Exercise 1.19. A test is based on 30 points. André got $\frac{3}{5}$ of the points. How many points did he get?

Exercise 1.20. One class contains 32 students. Among them, $\frac{3}{8}$ come to school by bus. How many students does this represent?

Exercise 1.21. In a street, there are 20 houses and $\frac{3}{4}$ of them have satellite TV. How many houses does this represent?

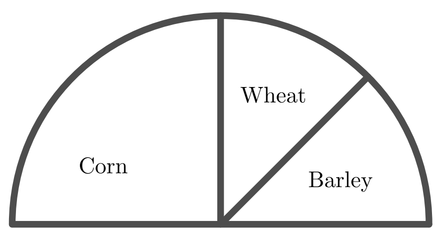
Exercise 1.22. A 20'000 liter tank is filled $\frac{3}{5}$ full. How many liters does it contain?

Exercise 1.23. In this semi-circular diagram, we see the distribution of plants grown by Mr. Eugene over 140 hectares. How many hectares are occupied by:

a) corn?

b) wheat?

c) barley?



Exercise 1.24. During a basketball tournament, Joachim shot 8 free throws and scored 6. In the same tournament, Tony took 13 free throws and scored 9. Which one is more skillful?

Exercise 1.25. A couple wins 600 francs in the lottery. They decide to divide this amount as follows:

— a third will go to the bank into their savings account;

— a quarter will be dedicated to a great restaurant dinner;

— two sixths will be used to repair their daughter's bikes.

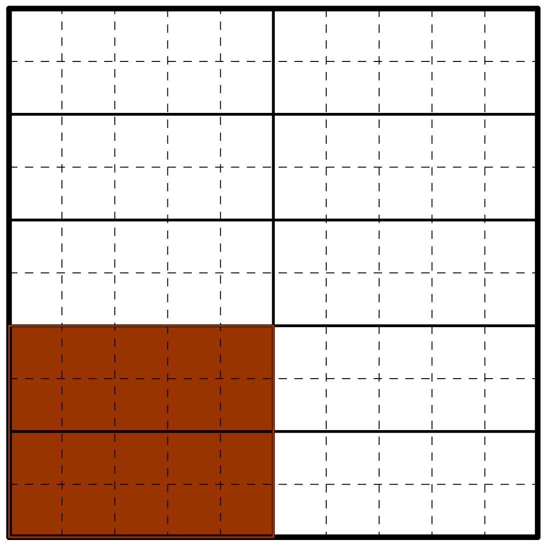
What will be left of the money?

Exercise 1.26. During the 50-minute math class, Julie spent half the time chatting, a quarter of the time giggling, a sixth of the time sleeping, a thirtieth of the time throwing paper balls and the rest of the time working. How long did Julie work?

1.4.7 Percentage

Definition. A *percentage* is a way of expressing a ratio using a fraction whose denominator is 100. Usually, this number is followed by the symbol %.

Example. The fraction $\frac{2}{10}$ is equal to $\frac{20}{100} = 20\% = 0,2$.



Compute a percentage

Example. 56 people out of 400 have blue eyes.

This proportion is expressed using the fraction $\frac{56}{400}$.

As

$$\frac{56}{400} = \frac{14}{100},$$

it means that 14% of the 400 people have blue eyes.

Applying a percentage

Example.

1. If a meeting of 120 people has 15% women in it, that means there are 18 women in that meeting. Indeed, by the rule of three, we have

$$\begin{array}{ccc} 120 & \to & 100\% \\ x & \to & 15\% \end{array}$$

and then

$$120 \cdot 15\% = 120 \cdot \frac{15}{100} = 18.$$

2. There are 36 women in a meeting. They represent 30% of the members. Therefore, the meeting is formed of 120 individuals. In fact, by the rule of three, we have

$$\begin{array}{ccc} 36 & \to & 30\% \\ x & \to & 100\% \end{array}$$

and then

$$\frac{36}{30\%} = 36 \cdot \frac{100}{30} = 120.$$

Exercise 1.27. Compute

a) 18% of 350.

b) 32% of 500.

c) $20,6\%$ of 1'200.

Exercise 1.28. A salesman offers a 15% discount on all items in his store. How much would you pay for a television whose price was originally 1'564 francs?

Exercise 1.29. The price of a car (which is 34'500 francs) is reduced by 7%, then by another 4%. What is the new price of the car?

Exercise 1.30. The initial price of a mobile phone is 205 francs. After a discount, it is sold for 161,95 francs. What was the percentage of the discount given by the shopkeeper?

Exercise 1.31. During a liquidation, a store will give an initial 50% discount on certain items, and then an additional 20% off the discounted price. What is the price paid for an item originally displayed at 50 francs? What is the total discount given in %?

Exercise 1.32. Steve paid his mechanic 202,50 francs after a 17% discount. What was the price before the discount?

Exercise 1.33. After an increase of 12%, an item is sold for 1'377,50 francs. What was the price of the item before the increase?

Exercise 1.34. A computer is sold for 1'615 francs. What was its initial price knowing that the discount represents 15% of the initial price?

Exercise 1.35. For a car costing 20'000 francs, is it better to choose

— a 10% discount?

— a 6% discount followed by a 10% discount?

— an 8% discount followed by another 8% discount?

— a 16% discount?

1.5 Real numbers

Lastly, there are numbers that cannot be written as a fraction. They're called *irrational numbers*. Discovered by the Greeks (who had trouble accepting their existence), they appear for example when studying the length of the sides of a triangle or the perimeter of a circle. When combined with rational numbers, they form the set of *real numbers*.

$$\mathbb{R} = \mathbb{Q} \cup \left\{ \sqrt{2}; \sqrt{3}; \sqrt{5}; \sqrt[3]{4}; \pi; e; \frac{1+\sqrt{5}}{2}; ... \right\}.$$

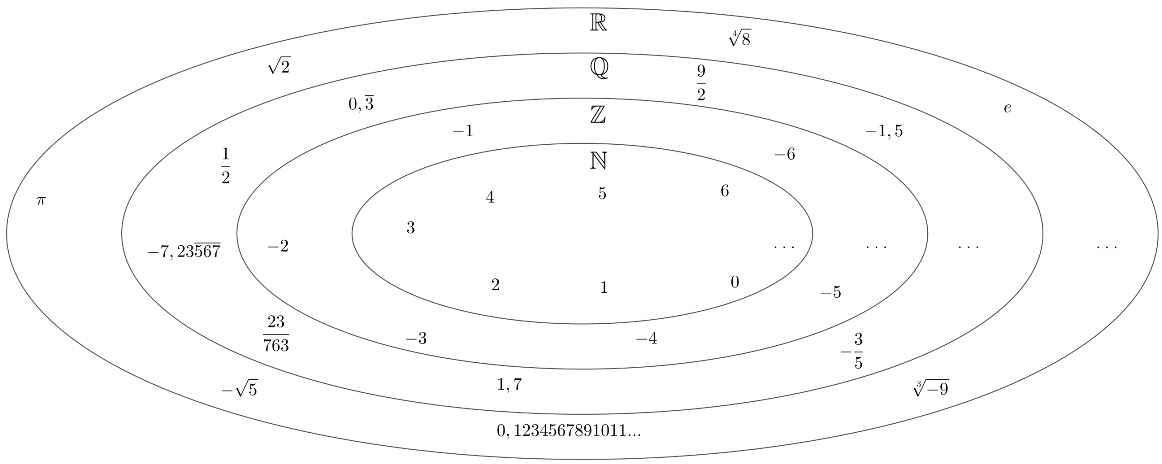


Figure 1.8: Set of real numbers.

We have the following set inclusions:

$$\boxed{\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.}$$

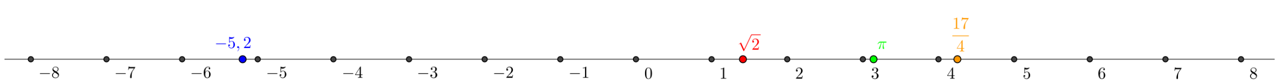


Figure 1.9: Numerical line.

Exercise 1.36. Tick the appropriate sets to which the numbers below belong.

Numbers	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
$-5,63$				
$9,\overline{2}$				
2π				
$-\dfrac{15}{3}$				
$\sqrt{2}$				
$\sqrt{9}$				

Exercise 1.37. Determine whether the statements below are true or false.

a) $-300 \in \mathbb{R}$ b) $\dfrac{20}{4} \notin \mathbb{N}$

c) $\dfrac{\sqrt{2}}{2} \in \mathbb{Q}$ d) $-\dfrac{13}{7} \notin \mathbb{Z}$

e) $-3,57 \in \mathbb{Z}$ f) $5 \notin \mathbb{R}$

Exercise 1.38. Represent the four sets of numbers in a Venn diagram, then place the following numbers in the correct area of that diagram.

a) $\sqrt{36}$ b) $-12,47$

c) $\dfrac{3}{4}$ d) π

e) $2,3 \cdot 10^{12}$ f) $-1'000'000$

g) $\sqrt{2}$ h) $5,12\overline{34}$

i) $-\dfrac{15}{3}$ j) $0,00000345$

1.6 Powers and roots

1.6.1 Definition

Definition. *the power* of a number a is defined by $a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ factors}}$.

We have many ways to say it: a to the power n, a to the power of n, a to the n-th power, a raised to the power of n or a to the n. The letter n is called *the exponent*.

Example.

$$\begin{array}{lrcl} a^1 = a, & 3^1 & = & 3. \\ a^2 = a \cdot a, & (-2)^2 & = & (-2) \cdot (-2) = 4. \\ a^3 = a \cdot a \cdot a, & \left(\dfrac{2}{5}\right)^3 & = & \dfrac{2}{5} \cdot \dfrac{2}{5} \cdot \dfrac{2}{5} = \dfrac{2^3}{5^3} = \dfrac{8}{125}. \end{array}$$

Exercise 1.39. Compute without calculator.

$$\begin{array}{llll} \text{a)} & 3^3 & -3^3 & -(3)^3 \\ \text{b)} & (-2)^3 & -2^3 & -(-2)^3 \\ \text{c)} & (-2)^4 & -2^4 & -(-2)^4 \\ \text{d)} & \left(\dfrac{1}{2}\right)^2 & \left(\dfrac{1}{2}\right)^3 & \left(\dfrac{2}{3}\right)^2 \\ \text{e)} & \left(-\dfrac{3}{2}\right)^2 & -\left(\dfrac{3}{2}\right)^2 & \left(-\dfrac{3}{2}\right)^3 \end{array}$$

1.6.2 Properties of powers

Theorem. *If $a, b \in \mathbb{R}$ and $m, n \in \mathbb{N}^*$, then*

1. $a^m \cdot a^n = a^{m+n}$.
2. $\dfrac{a^m}{a^n} = a^{m-n}$.
3. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.
4. $(a \cdot b)^n = a^n \cdot b^n$.
5. $(a^n)^m = a^{n \cdot m}$.

Proof. We will limit ourselves to giving the idea of proof in the particular case where $a = 4$, $b = 7$, $m = 5$ and $n = 3$.

1. $4^5 \cdot 4^3 = (4 \cdot 4 \cdot 4 \cdot 4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4^8 = 4^{5+3}$.
2. $\dfrac{4^5}{4^3} = \dfrac{{}^1\cancel{4} \cdot {}^1\cancel{4} \cdot {}^1\cancel{4} \cdot 4 \cdot 4}{{}^1\cancel{4} \cdot {}^1\cancel{4} \cdot {}^1\cancel{4}} = 4^2 = 4^{5-3}$.
3. $\left(\dfrac{4}{7}\right)^3 = \dfrac{4}{7} \cdot \dfrac{4}{7} \cdot \dfrac{4}{7} = \dfrac{4 \cdot 4 \cdot 4}{7 \cdot 7 \cdot 7} = \dfrac{4^3}{7^3}$.
4. $(4 \cdot 7)^3 = (4 \cdot 7) \cdot (4 \cdot 7) \cdot (4 \cdot 7) = (4 \cdot 4 \cdot 4) \cdot (7 \cdot 7 \cdot 7) = 4^3 \cdot 7^3$.
5. $(4^3)^5 = 4^3 \cdot 4^3 \cdot 4^3 \cdot 4^3 \cdot 4^3 = 4^{3+3+3+3+3} = 4^{5 \cdot 3}$.

$\square$

Exercise 1.40. Without using a calculator, reduce the expressions below as much as possible. We ask for a result using exponents.

a) $5 \cdot 5^4 \cdot 5^2$ b) $6^9 : (6^2 \cdot 6^3)$

c) $(2^4)^2 \cdot 2^3$ d) $-2,5 \cdot (-2,5) \cdot (-2,5) \cdot (-2,5)$

e) $\dfrac{(-8)^{10}}{(-8)^8}$ f) $2^3 \cdot 2^6 \cdot (-2)^4 \cdot (-2)^5$

g) $(-2^4)^3$ h) $(-2)^3 \cdot (-3)^3 \cdot (-1)^{1234567}$

i) $\left(\left(\dfrac{1}{2}\right)^2\right)^3$ j) $\left(\dfrac{3}{5}\right)^4 : \left(\dfrac{2}{3}\right)^5$

Exercise 1.41. Find, if it exists, the value or values of x that verify the following equations.

a) $2^3 \cdot 2^x = 2^5$ b) $6^3 \cdot 6^x = 6^3$

c) $7^5 : 7^x = 7^2$ d) $(-2)^x = 8$

e) $-x^2 = -25$ f) $x^3 : x^1 = 16$

1.6.3 Negative and zero exponents

Theorem. *If* $a \in \mathbb{R}^*$ *and* $n \in \mathbb{N}$*, then*

1. $a^0 = 1$.
2. $a^{-n} = \dfrac{1}{a^n}$.

Proof. The idea is to compute $\dfrac{a^n}{a^n}$, respectively $\dfrac{a^0}{a^n}$, in two different ways:

1. $\dfrac{a^n}{a^n} = \begin{cases} a^{n-n} = a^0 \\ 1 \end{cases}$.

2. $\dfrac{a^0}{a^n} = \begin{cases} a^{0-n} = a^{-n} \\ \dfrac{1}{a^n} \end{cases}$.

$\square$

Remark. These properties are clearly illustrated in the diagram below.

$$\begin{aligned} a^3 &= a \cdot a \cdot a \\ a^2 &= a \cdot a \\ a^1 &= a \\ a^0 &= 1 \\ a^{-1} &= \tfrac{1}{a} \\ a^{-2} &= \tfrac{1}{a^2} \\ a^{-3} &= \tfrac{1}{a^3} \end{aligned}$$

We notice that to get to the inferior line, we decrease the exponent by 1 (to the left of the equality sign) and divide by a (to the right of the equality sign).

Example.

1. $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.
2. $3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$.
3. $5^{-1} = \dfrac{1}{5}$.
4. $\left(\dfrac{2}{3}\right)^{-1} = \dfrac{1}{\frac{2}{3}} = 1 : \dfrac{2}{3} = 1 \cdot \dfrac{3}{2} = \dfrac{3}{2}$.

Exercise 1.42. Compute without a calculator. A solution as an integer or as an irreducible fraction is required.

a)	2^{-2}	$(-2)^{-2}$	-2^{-2}
b)	3^0	3^{-1}	-3^{-2}
c)	-5^0	-5^{-1}	$(-5)^{-1}$
d)	$\left(\dfrac{3}{2}\right)^{-1}$	$\left(\dfrac{3}{2}\right)^{-2}$	$\left(\dfrac{2}{3}\right)^{0}$

1.6.4 Roots

Definition. Let $a \in \mathbb{R}_+^*$ and $n \in \mathbb{N}^*$ be numbers. We say n*-th root* of a, denoted by $\sqrt[n]{a}$, the only positive number r such that $r^n = a$. In other words:

$$\boxed{r = \sqrt[n]{a} \iff r^n = a \text{ and } r \geq 0.}$$

The number a is called the *radicand*, the number n is called the index and $\sqrt[n]{\ }$ is called the *radical*.

In other words, looking for the n-th root of a is like wondering what number to the power n gives a.

Remark. The square root is written $\sqrt{a}$ instead of $\sqrt[2]{a}$.

Example.

1. $\sqrt{9} = 3$ because $3^2 = 9$.
2. $\sqrt[3]{8} = 2$ because $2^3 = 8$.

Definition. Let $a \in \mathbb{R}_-^*$ and $n \in \mathbb{N}^*$ be numbers.

— If $a < 0$ and n is odd, we define the n-th root by

$$\boxed{r = \sqrt[n]{a} \iff r^n = a.}$$

— If $a < 0$ and n is even, the n-th root of a doesn't exist.

Example.

1. $\sqrt{-9}$ doesn't exist. Indeed, any numbers squared is positive (or zero).
2. $\sqrt[3]{-8} = -2$ because $(-2)^3 = -8$.

Theorem. *Let a and b be two strictly positive real numbers, m, n and q strictly positive integers and p any integer. We have:*

1. $(\sqrt[n]{a})^n = a$.
2. $\sqrt[n]{a^n} = a$.
3. $\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$.
4. $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.
5. $(\sqrt[n]{a})^p = \sqrt[n]{a^p}$.
6. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[n \cdot m]{a}$.
7. $\sqrt[n \cdot q]{a^{n \cdot p}} = \sqrt[q]{a^p}$.
8. $\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.
9. $a^{\frac{1}{2}} = \sqrt{a}$.
10. $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Remark. We define $a^{\frac{1}{2}} = \sqrt{a}$ in order to satisfy the property $(a^m)^n = a^{mn}$. Indeed:

$$9^{\frac{1}{2}} = \left(3^2\right)^{\frac{1}{2}} = 3^{2 \cdot \frac{1}{2}} = 3 \Rightarrow \sqrt{9} = 3 = 9^{\frac{1}{2}}.$$

Example.

1. $(\sqrt{5})^2 = 5$.
2. $\sqrt[3]{5^3} = 5$.
3. $\sqrt[3]{3} \cdot \sqrt[3]{9} = \sqrt[3]{3 \cdot 9} = \sqrt[3]{27} = 3$.
4. $\sqrt{\dfrac{7}{4}} = \dfrac{\sqrt{7}}{\sqrt{4}}$.
5. $(\sqrt[3]{5})^2 = \sqrt[3]{5^2} = \sqrt[3]{25}$.
6. $\sqrt{\sqrt[3]{5}} = \sqrt[6]{5}$.
7. $\sqrt[6]{3^4} = \sqrt[3]{3^2} = \sqrt[3]{9}$.
8. $\sqrt{16+9} = \sqrt{25} = 5 \neq \sqrt{16} + \sqrt{9} = 4 + 3 = 7$.
9. $64^{\frac{1}{2}} = \sqrt{64} = 8$.
10. $27^{\frac{1}{3}} = \sqrt[3]{27} = 3$.

Exercise 1.43. Compute.

a) $\sqrt{0}$	b) $\sqrt{16}$
c) $\sqrt{36}$	d) $\sqrt{625}$
e) $\sqrt[3]{1000}$	f) $\sqrt[3]{216}$
g) $\sqrt[3]{-64}$	h) $\sqrt[3]{343}$
i) $\sqrt[3]{729}$	j) $\sqrt[4]{2401}$
k) $\sqrt[4]{-625}$	l) $\sqrt[5]{-32}$

Exercise 1.44. Write the following expressions using roots and simplify.

a) $9^{\frac{1}{2}}$	b) $100^{\frac{1}{2}}$
c) $(-16)^{\frac{1}{2}}$	d) $8^{\frac{1}{3}}$
e) $1024^{\frac{1}{10}}$	f) $0^{\frac{1}{5}}$
g) $25^{0,5}$	h) $\left(\frac{1}{16}\right)^{\frac{1}{2}}$
i) $36^{-\frac{1}{2}}$	j) $64^{-\frac{1}{3}}$

1.7 Solutions

Exercise 1.1.

a) 37.

b) 35.

c) 105.

d) 12.

e) For example 1 and 42, 2 and 21 or 3 and 14.

Exercise 1.2.

a) 19	b) 20
c) 41	d) 60
e) 120	f) Undetermined
g) 96	h) Impossible
i) 192	j) 162

Exercise 1.3.

a) $5 \cdot (18 + 4) = 110$	b) $(5 + 3) \cdot (1 + 1) = 16$
c) $80 + 40 : 2 = 100$	d) $(2 + 2) \cdot (2 + 2 \cdot 2) = 24$
e) $(9 - 9) \cdot 9 + 9 = 9$	f) $(3 \cdot 2 + 5) \cdot 6 = 66$
g) $30 : (2 + 8) = 3$	h) $(5 - 2) \cdot 9 - 7 = 20$
i) $(100 - 1) \cdot 100 - 1 = 9899$	j) $3 \cdot (3 + 3 \cdot 3) + 3 - 3 = 36$ or $(3 \cdot 3 + 3) \cdot 3 + 3 - 3$

Exercise 1.4.

a) -35	b) 3
c) -5	d) 3
e) 240	f) -231
g) 48	h) 9

Exercise 1.5.

a) 13	b) 1
c) 16	d) -35
e) -11	f) -46

Exercise 1.6.

a) 7	b) -2
c) -44	d) 9
e) 90	f) Impossible
g) Undetermined	h) Undetermined
i) 81	j) Undetermined
k) -5	l) 2
m) 2	n) -1

Exercise 1.7.

a) 1	b) 0
c) 0	d) -1
e) -6	f) -6

Exercise 1.8.

a) 10	b) 5
c) 11	d) 11
e) 17	f) 10
g) 3	h) 3
i) 3	j) -3
k) 4	l) -15

Exercise 1.9.

a) Not possible.

b) 0.

c) 56.

d) 6.

Exercise 1.10. In the first situation, Alice eats more pizza, while Alain eats more in the second.

Exercise 1.11.

a) $\dfrac{7}{5} = \dfrac{35}{25}$	b) $\dfrac{16}{18} = \dfrac{64}{72}$
c) $\dfrac{8}{20} = \dfrac{40}{100}$	d) $\dfrac{3}{14} = \dfrac{9}{42}$
e) $\dfrac{24}{11} = \dfrac{264}{121}$	f) $\dfrac{4}{8} = \dfrac{5}{10}$
g) $\dfrac{30}{24} = \dfrac{40}{32}$	h) $\dfrac{27}{63} = \dfrac{33}{77}$
i) $\dfrac{27}{36} = \dfrac{21}{28}$	j) $\dfrac{21}{15} = \dfrac{35}{25}$

Exercise 1.12.

a) $\frac{3}{4}$ b) $\frac{7}{3}$

c) $\frac{9}{7}$ d) $\frac{3}{4}$

e) $\frac{11}{14}$ f) $\frac{5}{3}$

g) $\frac{14}{13}$ h) $\frac{3}{5}$

i) $\frac{8}{9}$ j) $\frac{3}{4}$

k) $\frac{5}{8}$ l) $\frac{1}{13}$

m) $\frac{2}{5}$ n) $\frac{5}{48}$

o) $\frac{8}{75}$ p) $\frac{40}{3}$

q) $\frac{11}{3}$ r) $\frac{4}{15}$

s) 12 t) $\frac{38}{27}$

Exercise 1.13.

a) For example $\frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \frac{30}{40}$ and $\frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{25}{30} = \frac{50}{60}$.

b)

a) $\frac{90}{120}$ and $\frac{100}{120}$.

b) $\frac{120}{160}$ and $\frac{120}{144}$.

c) $\frac{9}{12}$ and $\frac{10}{12}$.

d) $\frac{75}{100}$ and not possible.

e) $\frac{105}{140}$ and $\frac{105}{126}$.

f) $\frac{45}{60}$ and $\frac{50}{60}$.

Exercise 1.14.

a) $\frac{5}{8}$ b) $\frac{7}{15}$

c) $\frac{11}{18}$ d) $\frac{11}{36}$

Exercise 1.15.

a) 1

b) $\frac{1}{6}$

c) $\frac{17}{6}$

d) 2

e) $\frac{155}{84}$

f) $\frac{634}{75}$

g) $\frac{39}{14}$

h) $\frac{13}{18}$

i) $\frac{25}{12}$

j) $\frac{1}{30}$

k) $\frac{13}{12}$

l) $\frac{19}{32}$

m) $\frac{1}{6}$

n) $\frac{23}{6}$

o) $\frac{19}{5}$

p) $\frac{761}{100}$

Exercise 1.16.

a) $\frac{7}{4}$

b) $\frac{3}{64}$

c) $\frac{7}{12}$

d) $\frac{3}{8}$

e) $\frac{5}{14}$

f) 75

g) 1

h) $\frac{7}{20}$

i) $-\frac{20}{7}$

j) $-\frac{14}{9}$

k) $-\frac{4}{27}$

l) $\frac{7}{4}$

m) 0

n) $\frac{1}{3}$

Exercise 1.17.

a) 4

b) $\frac{44}{21}$

c) $\frac{5}{4}$

d) $\frac{21}{5}$

e) $\frac{1}{4}$

f) 51

g) $\frac{3}{40}$

h) $\frac{96}{5}$

i) 10

j) $\frac{160}{9}$

k) $\frac{20}{27}$

l) $\frac{120}{121}$

Exercise 1.18.

a) $\dfrac{128}{17}$ b) $\dfrac{33}{56}$

c) $\dfrac{109}{30}$ d) $\dfrac{113}{60}$

e) 13 f) $-\dfrac{8}{13}$

g) $\dfrac{7}{6}$ h) $\dfrac{43}{5}$

Exercise 1.19. 18 points.

Exercise 1.20. 12 students.

Exercise 1.21. 15 houses.

Exercise 1.22. 12'000 liters.

Exercise 1.23.

a) 70 ha.
b) 35 ha.
c) 35 ha.

Exercise 1.24. Joachim was more skillful.

Exercise 1.25. 50 francs.

Exercise 1.26. 2 minutes and 30 seconds.

Exercise 1.27.

a) 63.
b) 160.
c) $247,2$.

Exercise 1.28. $1'329,4$ francs.

Exercise 1.29. $30'801,6$ francs.

Exercise 1.30. 21%.

Exercise 1.31. Price paid: 20 francs, Total discount: 60%.

Exercise 1.32. 244 francs.

Exercise 1.33. $1'229,90$ francs.

Exercise 1.34. Initial price: 1'900 francs.

Exercise 1.35. A 16% discount.

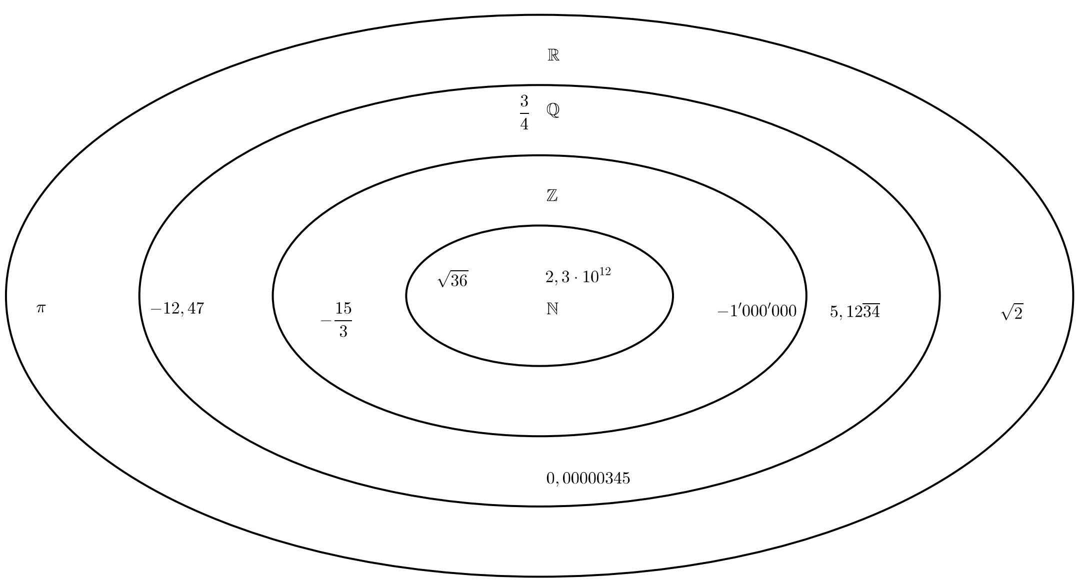
Exercise 1.36.

Numbers	$\mathbb{N}$	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$
$-5,63$			x	x
$9,\overline{2}$			x	x
2π				x
$-\dfrac{15}{3}$		x	x	x
$\sqrt{2}$				x
$\sqrt{9}$	x	x	x	x

Exercise 1.37.

a) True b) False
c) False d) True
e) False f) False

Exercise 1.38.



Exercise 1.39.

a)	27	-27	-27
b)	-8	-8	8
c)	16	-16	-16
d)	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{4}{9}$
e)	$\frac{9}{4}$	$-\frac{9}{4}$	$-\frac{27}{8}$

Exercise 1.40.

a) 5^7
b) 6^4
c) 2^{11}
d) $(-2,5)^4 = 2,5^4$
e) $(-8)^2 = 8^2$
f) $2^9 \cdot (-2)^9 = -2^{18}$
g) -2^{12}
h) -6^3
i) $\frac{1}{2^6}$
j) $\frac{3^9}{5^4 \cdot 2^5}$

Exercise 1.41.

a) $x = 2$
b) $x = 0$
c) $x = 3$
d) No solution
e) $x = 5$ and $x = -5$
f) $x = 4$ and $x = -4$

Exercise 1.42.

a)	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{4}$
b)	1	$\frac{1}{3}$	$-\frac{1}{9}$
c)	-1	$-\frac{1}{5}$	$-\frac{1}{5}$
d)	$\frac{2}{3}$	$\frac{4}{9}$	1

Exercise 1.43.

a) 0	b) 4
c) 6	d) 25
e) 10	f) 6
g) -4	h) 7
i) 9	j) 7
k) Not defined	l) -2

Exercise 1.44.

a) 3	b) 10
c) Not defined	d) 2
e) 2	f) 0
g) 5	h) $\dfrac{1}{4}$
i) $\dfrac{1}{6}$	j) $\dfrac{1}{4}$

1.8 Chapter objectives

At the end of this chapter, the student should be able to

1.1 ☐ Calculate an arithmetic expression using the order of operations.
1.2 ☐ Distinguish the different types of division involving 0.
1.3 ☐ Multiply or divide integers according to the sign rules.
1.4 ☐ Add or subtract integers.
1.5 ☐ Compute expressions with absolute values.
1.6 ☐ Expand a fraction.
1.7 ☐ Simplify a fraction as much as possible.
1.8 ☐ Add or subtract fractions.
1.9 ☐ Multiply or divide fractions.
1.10 ☐ Solve a problem involving fractions.
1.11 ☐ Solve a problem involving percentages.
1.12 ☐ Sort given numbers into the correct set.
1.13 ☐ Raise a number to an integer exponent (positive or negative).
1.14 ☐ Simplify an expression containing powers using properties.
1.15 ☐ Compute the n-th root of a number.
1.16 ☐ Master the vocabulary related to numbers and operations.

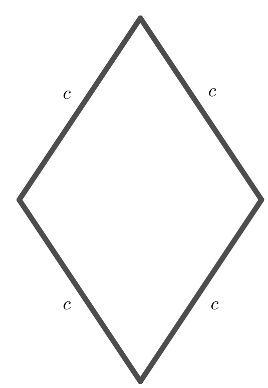
Chapter 2

Elementary algebra

2.1 Introduction

In mathematics, we often work with *letters*, which represent *numbers*. The purpose of the *elementary algebra* is to represent unknown numbers with letters in order to solve problems.

Example.
A rhombus has 4 equal sides of length c.
Its perimeter P is obtained by adding the lengths of its sides, so it is equal to :



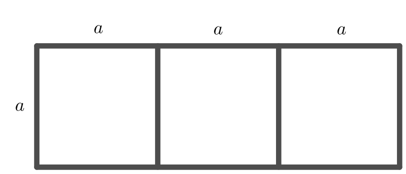
$$\begin{aligned} P &= c+c+c+c \\ &= 4\cdot c \\ &= 4c \end{aligned}$$

With this formula, we can avoid repeating the same process every time we have to calculate the perimeter of a rhombus.
For example, to calculate the perimeter of a rhombus whose side measures 12 cm, we replace c by 12 in the expression given by the formula $P = 4c$. The perimeter of this diamond is then equal to

$$P = 4 \cdot 12 = 48 \text{ cm}.$$

In this situation, we say that c is a *variable*.

Example.
A rectangle is formed by connecting three identical squares. The length of the side of each square is a.



The area of each square is equal to $a \cdot a$. As a result, the area A of the rectangle is equal to the sum of the areas of the three squares.
It's therefore equal to

$$A = a \cdot a + a \cdot a + a \cdot a = 3 \cdot a \cdot a = 3a^2.$$

2.2 Monomials and polynomials

2.2.1 Definition

Definition. An *algebraic expression* is an expression containing one or several variables.

Example.

1. $3x^8$ is a *monomial* because it has only one term. x is the *variable*, because it can be of any value. 3 is called the *coefficient* of x^8.
2. $3x^8 - 5y$ is a *binomial*, because it has two terms.
3. $5x^2 - 2x + 3$ is a *trinomial*, because it has three terms.

Definition. When the number of terms isn't specified, it's called a *polynomial*. The *degree* of a polynomial with respect to a variable is the greatest exponent observed with respect to the variable.

Example.

1. $5x^3 - 3x^2 + 2x - 9$ is a polynomial of degree 3.
2. $3x^4y - 2xy^5$ is a polynomial of degree 4 with respect to the variable x and of degree 5 with respect to y.

Definition. We call *like terms* the terms that have exactly the same variable.

Example.

1. $3x^2y^5$ and $-6x^2y^5$ are like terms.
2. $3x^2y^5$ and $-6x^5y^2$ are not like terms.

2.2.2 Operations on monomials

Addition and subtraction

It is only possible to add or subtract monomials if they are like terms. We add or subtract their coefficients and keep the variables.

Example.

1. $3x + 5x = (3 + 5)x = 8x$.
2. $6x + 3y - 2x + 8y = (6 - 2)x + (3 + 8)y = 4x + 11y$
3. $\frac{3}{4}x^3y^2 + \frac{1}{2}y^2x^3 = \left(\frac{3}{4} + \frac{1}{2}\right)x^3y^2 = \frac{5}{4}x^3y^2$

Exercise 2.1. Compute and reduce the like terms.

a) $2a + 8a - 12a$

b) $xyz + xyz$

c) $-3ab + 4ab + 3ab$

d) $3x - 2x + 5x - 2x + 4x - 3x$

e) $5b^2 - b^2 - 4b^2$

f) $ab - 3ab + 7ba - 10ab + ba$

g) $2ab - 3ab + 5ba - 14ab + ba$

h) $18ab^2 - 3a^2b - 8a^2b + 5ab^2$

i) $a^2x^3 - 5a^3x^2 - 6a^3x^2$

j) $-(-4a) + 6c - 7a$

k) $-(-5uv) - 10u^2v + uv - (-u^2v)$

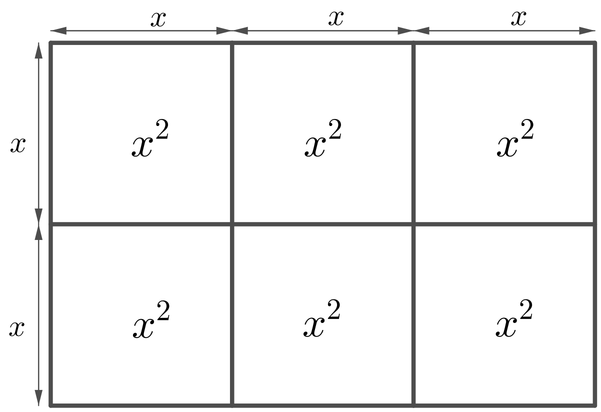
l) $2x^2y^3 - 3y^3x^2 + 5yxyxy$

Multiplication

When multiplying monomials, we multiply the coefficients between them and also the variables between them.

Example.

1. $3x \cdot 2x = (3 \cdot 2)(x \cdot x) = 6x^2$



2. $\frac{3}{4}x^2y^3z \cdot \frac{5}{2}x^3y = \left(\frac{3}{4} \cdot \frac{5}{2}\right)(x^2 \cdot x^3)(y^3 \cdot y)(z) = \frac{15}{8}x^5y^4z$

Exercise 2.2. Compute.

a) $x^5 \cdot x^4 \cdot x^3$
b) $2x^2 \cdot 4x^4$
c) $5x^2y \cdot (-2x^3y^2)$
d) $(-4ab^2c^3) \cdot (-3a^5b^4c^3)$
e) $\left(\frac{2}{3}x^5y^6\right) \cdot 3xy^3$
f) $\left(\frac{7}{6}x^2y^5z\right) \cdot \left(\frac{9}{14}x^5yz^2\right)$
g) $(3x^2y)^2$
h) $\left(\frac{5}{7}a^2b^3c^4\right)^2$
i) $(4x^3y^2)^2 \cdot (2xy^4)^3$
j) $(5x^2y) \cdot (4x^3y^4)^2 \cdot (3x^5y^6)^3$

Exercise 2.3. Compute and reduce.

a) $(2b)^3 - b^3 + 2b^3$
b) $(3x^2)^2 + (2x)^4$
c) $2x \cdot 3x^2 - 5x^3 + (3x)^3$
d) $(2a)^2 \cdot 4b^2 + (5ab)^2 - (-2ab)^2$

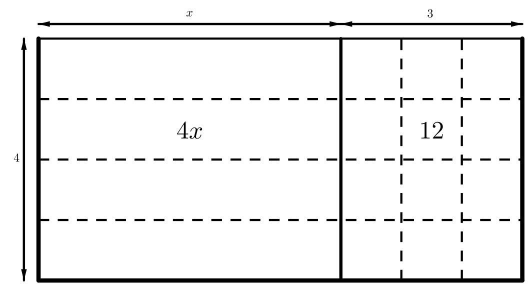
2.2.3 Operations on polynomials

As seen earlier, it is only possible to add or subtract two terms together if they are like terms. When multiplying two polynomials, we have to distribute.

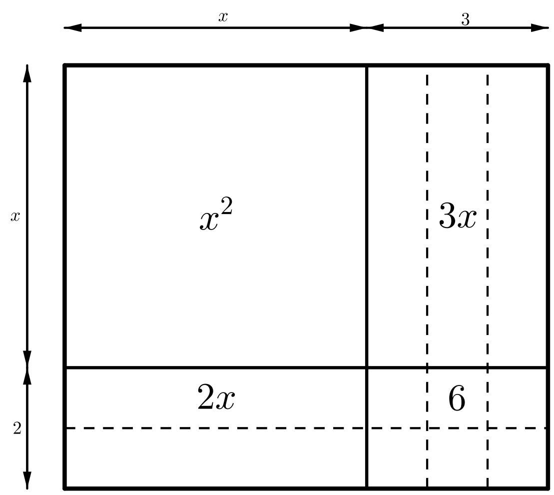
Example.

1. $(3x^2 - 2x + 7) + (x^2 + 3x) = 3x^2 - 2x + 7 + x^2 + 3x = 4x^2 + x + 7$.
2. $(3x^2 - 2x + 7) - (x^2 - x^3 + 3x) = 3x^2 - 2x + 7 - x^2 + x^3 - 3x = x^3 + 2x^2 - 5x + 7$.

3. $4 \cdot (x+3) = 4 \cdot x + 4 \cdot 3 = 4x + 12.$



4. $(x+2) \cdot (x+3) = x \cdot x + x \cdot 3 + 2 \cdot x + 2 \cdot 3 = x^2 + 3x + 2x + 6 = x^2 + 5x + 6.$



5. $(3x-2) \cdot (x^2+3) = 3x \cdot x^2 + 3x \cdot 3 - 2 \cdot x^2 - 2 \cdot 3 = 3x^3 + 9x - 2x^2 - 6 = 3x^3 - 2x^2 + 9x - 6.$

Exercise 2.4. Reduce as much as possible the following expressions.

a) $3(5a+2)$ b) $-5(a-3)$

c) $\frac{1}{2}z(4+2z)$ d) $3x\left(\frac{2}{3}x - \frac{1}{6}\right)$

e) $-3(x-2y+3z)$ f) $(x+3)(x+5)$

g) $(x-2)(x+1)$ h) $(3z-1)(4+2z)$

Exercise 2.5. Reduce as much as possible the following expressions.

a) $(x^4 - x^3 + 2) + (x^2 - 2x + 5)$ b) $(2y)^5 \cdot (2z)^6$

c) $4x^2 + 3y - (6x^2 - 2y)$ d) $(x^2 - 2x - 3) - (x^2 + 4x + 9)$

e) $2(x+3) - 3(x-1)$ f) $4x(x-3) - (x-7)$

g) $18x - [7x - (8x - y)]$ h) $5x - [x - (5x - 3)]$

i) $4x - \{2x - [3y - (5x - 4y) + 3x]\} - 2y$ j) $25x - \{13x - [24x - (5x + 3y) - (7x - y)] + (24x - 2y)\}$

Exercise 2.6. Reduce as much as possible the following expressions.

a) $3x(x-4)(x+5)$ b) $-2x(x-3)(4-2x^2)$
c) $x\cdot(x+1)\cdot(x^2-x-1)$ d) $(2a+1)\cdot(3a-1)\cdot(2a-3)$
e) $5x^3-\{4x+3x[2x^2-3(x-5)]\}$ f) $-3x^3\{9x^2-[3x^3-2x(4x+1)]\}$
g) $-2(a+c)+3[(b-c)+3(c-a)]$ h) $(3x-1)(x+2)-(2x+5)(x-1)$

Exercise 2.7. Evaluate the following polynomials when $p=-2$, $q=4$ and $r=-5$.

a) $-3(p+5q)$ b) $\dfrac{q+r}{q+p}$

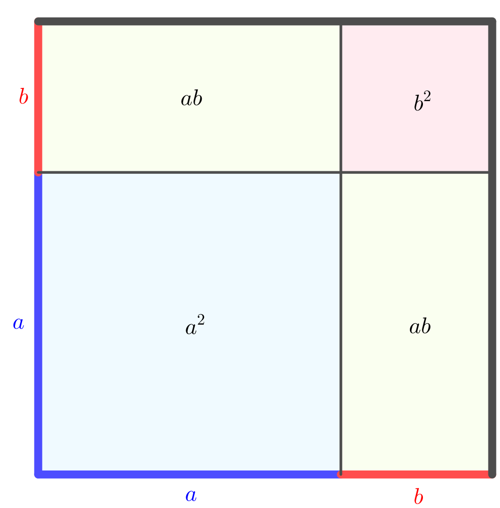
2.2.4 Remarkable identities

Theorem. *If a and b are two monomials, then*

1. $\boxed{(a+b)^2=a^2+2ab+b^2}$
2. $\boxed{(a-b)^2=a^2-2ab+b^2}$
3. $\boxed{(a+b)(a-b)=a^2-b^2}$
4. $\boxed{(a+b)^3=a^3+3a^2b+3ab^2+b^3}$
5. $\boxed{(a-b)^3=a^3-3a^2b+3ab^2-b^3}$

Proof. Let a and b be two monomials.
We have

1.
$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a^2+ab+ba+b^2\\ &= a^2+2ab+b^2.\end{aligned}$$



2.
$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-ab-ba+b^2\\ &= a^2-2ab+b^2.\end{aligned}$$

3.
$$\begin{aligned}(a+b)(a-b) &= a^2-ab+ba-b^2\\ &= a^2-b^2.\end{aligned}$$

4.
$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+a^2b+2a^2b+2ab^2+b^2a+b^3\\ &= a^3+3a^2b+3ab^2+b^3.\end{aligned}$$

5.
$$\begin{aligned}(a-b)^3 &= (a-b)^2(a-b)\\ &= (a^2-2ab+b^2)(a-b)\\ &= a^3-a^2b-2a^2b+2ab^2+b^2a-b^3\\ &= a^3-3a^2b+3ab^2-b^3.\end{aligned}$$

$\square$

Example.

1.
$$\begin{aligned}(5x+3y)^2 &= (5x)^2+2\cdot 5x\cdot 3y+(3y)^2\\ &= 25x^2+30xy+9y^2.\end{aligned}$$

2.
$$\begin{aligned}(4x^3-y^2)^2 &= (4x^3)^2-2\cdot 4x^3\cdot y^2+(y^2)^2\\ &= 16x^6-8x^3y^2+y^4.\end{aligned}$$

3.
$$\begin{aligned}(9x^4+2x^3)(9x^4-2x^3) &= (9x^4)^2-(2x^3)^2\\ &= 81x^8-4x^6.\end{aligned}$$

4.
$$\begin{aligned}(4x+5y)^3 &= (4x)^3+3\cdot(4x)^2\cdot 5y+3\cdot 4x\cdot(5y)^2+(5y)^3\\ &= 64x^3+3\cdot 16x^2\cdot 5y+3\cdot 4x\cdot 25y^2+125y^3\\ &= 64x^3+240x^2y+300xy^2+125y^3.\end{aligned}$$

5.
$$\begin{aligned}(5x^3-2y)^3 &= (5x^3)^3-3\cdot(5x^3)^2\cdot 2y+3\cdot 5x^3\cdot(2y)^2-(2y)^3\\ &= 125x^9-3\cdot 25x^6\cdot 2y+3\cdot 5x^3\cdot 4y^2-8y^3\\ &= 125x^9-150x^6y+60x^3y^2-8y^3.\end{aligned}$$

Exercise 2.8. Compute using remarkable identities.

a) $(x+1)^2$ b) $(x-3)^2$

c) $(x-6)(x+6)$ d) $(x+5)^2$

e) $(x-7)^2$ f) $(x-2)(x+2)$

g) $(-x+2)^2$ h) $(x-7)(x+7)$

i) $(2x-4)(2x+4)$ j) $(4m^2-9n^2)^2$

k) $(3x+6)(3x-6)$ l) $\left(\frac{2}{7}x-\frac{3}{2}y\right)^2$

m) $(xy^2z-5)(xy^2z+5)$ n) $\left(x-\frac{2}{3}\right)\left(x+\frac{2}{3}\right)$

Exercise 2.9. Fill in the gaps.

a) $9x^2 - 24x + \ ...\ = (\ ...\ - \ ...\ )^2$ b) $64x^2 + \ ...\ \cdot x + \dfrac{1}{9} = (\ ...\ + \ ...\ )^2$

c) $\left(\dfrac{1}{3}x + \ ...\ \right)^2 = \ ...\ + 4x + \ ...$ d) $\left(\dfrac{1}{5}x + \ ...\ \right)^2 = \ ...\ + \dfrac{3}{10}x + \ ...$

Exercise 2.10. Compute using remarkable identities.

a) $(x+1)^3$ b) $(x^2-1)^3$
c) $(2x+3y)^3$ d) $(3x-8)^3$
e) $(4x^2-3x^3)^3$ f) $(3x+2y^2)^3$

2.3 Factorization

By expanding $2x(x-y)$, we get $2x^2-2xy$. *Factoring* $2x^2-2xy$ consists of retrieving $2x(x-y)$.
In other words:

$$\boxed{2x(x-y) \underset{\text{Factorize}}{\overset{\text{Expand}}{\rightleftarrows}} 2x^2-2xy}$$

Depending on the case, different factorization methods will be used.

Example.

1. **Factoring out common factors**

   Let's factorize $8x^3y^2 - 12x^2y^3$.

   Since both monomials are multiples of $4x^2y^2$, it's possible to *factor out* this term:

   $$8x^3y^2 - 12x^2y^3 = 4x^2y^2(2x-3y).$$

   To check this, you just have to expand the term on the right.

2. **Factoring out a parenthesis**

   Let's factorize $2(x-y)^2 + 4(x-y)$.

   This expression can be seen as the sum of two terms, with each term being a multiple of $2(x-y)$. It is therefore possible to factor out $2(x-y)$:

   $$2(x-y)^2 + 4(x-y) = 2(x-y)[(x-y)+2] = 2(x-y)(x-y+2).$$

3. **Factoring by grouping**

   Let's factorize $2x + 2y + xz + yz$.

   It is not possible to factor out a term, because the monomials composing the polynomial have no common divisor. However, it is possible to factor out 2 from the first two terms and z from the last two terms, to get back to the previous case:

   $$2x + 2y + xz + yz = 2(x+y) + z(x+y) = (x+y)(2+z).$$

4. **Remarkable identities**

Let's factorize $9x^2 + 24xy + 16y^2$. This trinomial being of the form $a^2 + 2ab + b^2$, it could be written as $(a + b)^2$:

$$\underbrace{9x^2}_{=a^2} + \underbrace{24xy}_{=2ab} + \underbrace{16y^2}_{=b^2} = (\underbrace{3x}_{=a} + \underbrace{4y}_{=b})^2.$$

5. **Remarkable identities**

Concerning the binomial $25x^2 - 9y^2$, it's of the form $a^2 - b^2$. We therefore have

$$\underbrace{25x^2}_{=a^2} - \underbrace{9y^2}_{=b^2} = (\underbrace{5x}_{=a} + \underbrace{3y}_{=b})(\underbrace{5x}_{=a} - \underbrace{3y}_{=b}).$$

6. **Factoring a second degree trinomial**

If we expand $(x + 2)(x + 3)$, we get

$$(x + 2)(x + 3) = x^2 + 3x + 2x + 6 = x^2 + 5x + 6.$$

Factoring $x^2 + 5x + 6$ is to write it as the product of the two parentheses above.

We observe that:

— since the trinomial has the term x^2, each parenthesis must contain x;
— each parenthesis contains a binomial of the form $x + \text{Number}$;
— the $5x$ term was obtained by calculating $3x + 2x$;
— the number 6 was obtained by calculating $2 \cdot 3$.

In other words, the factorization will be of the form $(x + \text{Number}_1)(x + \text{Number}_2)$. The product of these two numbers must give 6 and their sum 5.

7. **Factoring a second degree trinomial**

Let's factorize $x^2 + 7x + 12$.

As $4 \cdot 3 = 12$ and $4 + 3 = 7$, we have

$$x^2 + 7x + 12 = (x + 4)(x + 3).$$

8. **Factoring a second degree trinomial**

Let's factorize $x^2 - 8x + 15$.

As $(-5) \cdot (-3) = 15$ and $-5 - 3 = -8$, we have

$$x^2 - 8x + 15 = (x - 5)(x - 3).$$

9. **Factoring a second degree trinomial**

   Let's factorize $x^2 - 2x - 24$.

   As $(-6) \cdot 4 = 24$ and $-6 + 4 = -2$, we have

$$x^2 - 2x - 24 = (x-6)(x+4).$$

10. **A mix**

    Let's factorize $2x^3 - 4x^2 - 16x$.

    We have

$$2x^3 - 4x^2 - 16x = 2x(x^2 - 2x - 8) = 2x(x-4)(x+2).$$

11. **A mix**

    Let's factorize $2x^4 - 2y^4$.

    We have

$$2x^4 - 2y^4 = 2(x^4 - y^4) = 2(x^2 + y^2)(x^2 - y^2) = 2(x^2 + y^2)(x+y)(x-y).$$

Exercise 2.11. Factor out the greatest common factor from the following expressions.

a) $2a + 2b$
b) $rs + 4st$
c) $12a + 15b - 9c$
d) $4u^2 - 2uv$
e) $10xy + 15xy^2$
f) $9b - 30abc + 3bc$
g) $4a^4 - 8a^3 + 20a^2 - 4a$
h) $3a^2b^2 - 6a^2b$
i) $3x^2y^3 - 9x^3y^2$
j) $16x^5y^2 + 8x^3y^3$
k) $15x^3y^4 - 25x^4y^2 + 10x^6y^4$
l) $121r^3s^4 + 77r^2s^4 - 55r^4s^3$

Exercise 2.12. Factor out the greatest common factor from the following expressions.

a) $n(x-y) - (x-y)$
b) $(4a-5b)(3p-2q) - (a+5b)(3p-2q)$
c) $r(a-2) + r^2(a-2) - r^3(a-2)$
d) $(x+1)(x-y) - (x-3)(x-y) - (x+2)(x-y)$
e) $(4x-3)^2 - (4x-3)(9x-9)$
f) $17(x-2) - 34(-2+x) + 85(x-2)^2$
g) $(5x+2y) - 2x(2y+5x)^2 + 7(5x+2y)$
h) $(2x-1)^2 - 3(2x-1)(x+2) + (x+4)(2x-1)$

Exercise 2.13. Factorize by grouping.

a) $2ax - 6bx + ay - 3by$
b) $4ax + 2bx - 6ay - 3by$
c) $3x^3 + 3x^2 - 27x - 27$
d) $5x^3 + 10x^2 - 20x - 40$
e) $3x^2 + 2xy + 6x + 4y$
f) $2ay^2 - axy + 6xy - 3x^2$
g) $1 - x + x^2 - x^3 + x^4 - x^5$
h) $ax - bx + 2x - ay + by - 2y$

Exercise 2.14. Factorize using remarkable identities.

a) $x^2 + 10x + 25$
b) $x^2 + 16x + 64$
c) $x^2 - 14x + 49$
d) $4x^2 + 4x + 1$
e) $9x^2 - 4$
f) $x^2 - 25$

Exercise 2.15. Factorize as much as possible the following trinomials.

a) $x^2 + 3x + 2$ b) $x^2 + 5x + 6$
c) $x^2 + 2x - 3$ d) $x^2 + 7x + 12$
e) $x^2 + x - 30$ f) $x^2 + 15x + 56$
g) $x^2 - 9x + 20$ h) $x^2 + x - 56$
i) $x^2 + x - 12$ j) $x^2 - 14x + 48$

Exercise 2.16. Factorize as much as possible.

a) $x^3 + 2x^2y + xy^2$ b) $x^3y - xy^3$
c) $x^4 - 25x^2$ d) $4x^3 + 4x^2 + x$
e) $81x^{13} - 72x^{12} + 16x^{11}$ f) $16x^{15} + 72x^{14} + 81x^{13}$
g) $x^8 - 256$ h) $x^5 - 81x$
i) $2x^3 - 22x^2 + 20x$ j) $32x^5 - 162x$

2.4 Algebraic fractions

2.4.1 Definition

Definition. An *algebraic fraction* is a quotient of two polynomials.

Example. $\dfrac{x+y}{2x+3y}$ and $\dfrac{x^2}{5x^2 - 11y^3}$ are two algebraic fractions.

2.4.2 Operations on algebraic fractions

The operations on algebraic fractions are identical to those on standard fractions:

1. Expanding an algebraic fraction consists in multiplying its numerator and denominator by the same polynomial.
2. Simplify an algebraic fraction consists in dividing its numerator and denominator by the same polynomial.
3. The product of two algebraic fractions is an algebraic fraction where the numerator is the product of the numerators and the denominator is the product of the denominators.
4. Dividing an algebraic fraction by another is as simple as multiplying the first by the inverse of the second.
5. Adding two algebraic fractions requires amplifying them in order to get a common denominator and adding the resulting numerators.

Example.

1. **Expand** $\dfrac{2x}{x-y}$ by $x+y$:

$$\frac{2x}{x-y}=\frac{2x}{x-y}\cdot\frac{x+y}{x+y}=\frac{2x(x+y)}{(x-y)(x+y)}=\frac{2x^2+2xy}{x^2-y^2}.$$

2. **Simplify** $\dfrac{x^2-9}{x^2+4x+3}$:

$$\frac{x^2-9}{x^2+4x+3}=\frac{\cancel{(x+3)}(x-3)}{(x+1)\cancel{(x+3)}}=\frac{x-3}{x+1}.$$

3. **Product of two algebraic fractions**

$$\begin{aligned}\frac{x+2}{x^2+8x+16}\cdot\frac{x^2+7x+12}{x+3} &= \frac{x+2}{(x+4)^2}\cdot\frac{(x+4)(x+3)}{x+3}\\ &= \frac{(x+2)\cancel{(x+4)}\cancel{(x+3)}}{(x+4)^{\cancel{2}}\cancel{(x+3)}}\\ &= \frac{x+2}{x+4}.\end{aligned}$$

4. **Quotient of two algebraic fractions**

$$\begin{aligned}\frac{x^2-4}{2x+6}:\frac{x-2}{x+3} &= \frac{x^2-4}{2x+6}\cdot\frac{x+3}{x-2}\\ &= \frac{(x+2)(x-2)}{2(x+3)}\cdot\frac{x+3}{x-2}\\ &= \frac{(x+2)\cancel{(x-2)}\cancel{(x+3)}}{2\cancel{(x+3)}\cancel{(x-2)}}\\ &= \frac{x+2}{2}.\end{aligned}$$

5. **Sum of two algebraic fractions**

$$\begin{aligned}\frac{3}{x+y}-\frac{2}{x-y} &= \frac{3(x-y)}{(x+y)(x-y)}-\frac{2(x+y)}{(x-y)(x+y)}\\ &= \frac{3(x-y)-2(x+y)}{(x+y)(x-y)}\\ &= \frac{3x-3y-2x-2y}{(x+y)(x-y)}\\ &= \frac{x-5y}{(x+y)(x-y)}.\end{aligned}$$

Remark.

1. Pay attention to the "$-$" sign placed before an algebraic fraction. This sign changes all the signs of the numerator. Indeed, we have for instance

$$\begin{aligned}\frac{2x}{x+y}-\frac{3x-4y}{x+y} &= \frac{2x}{x+y}+(-1)\cdot\frac{3x-4y}{x+y}\\ &= \frac{2x}{x+y}+\frac{-3x+4y}{x+y}\\ &= \frac{2x-3x+4y}{x+y}\\ &= \frac{-x+4y}{x+y}.\end{aligned}$$

2. The simplification

$$\frac{x^{\cancel{2}}+1}{4\cancel{x}+3}=\frac{x+1}{4+3}=\frac{x+1}{7}$$

is wrong !

To understand this, we just need a counter-example.

By assuming $x=2$, we have

$$\frac{2^2+1}{4\cdot 2+3}=\frac{5}{11}$$

and

$$\frac{2+1}{7}=\frac{3}{7}.$$

Exercise 2.17. Expand the fractions to get the denominator indicated after the semicolon.

a) $\dfrac{5x}{8y}$; $48x^2y^2$ b) $\dfrac{3a+1}{a+1}$; a^2-1

c) $\dfrac{2a+5}{a-5}$; $a^2-10a+25$ d) $\dfrac{x-3y}{3x-y}$; $9x^2-y^2$

Exercise 2.18. Simplify as much as possible.

a) $\dfrac{15a^3b^2}{20a^2b^4}$ b) $\dfrac{-18a^3xy^2}{42ax^2y^3}$

c) $\dfrac{3a^3b^2x^2y^7}{-121b^2x^5y^2}$ d) $\dfrac{(a+b)^2(a-b)}{(a-b)^2(a+b)}$

Exercise 2.19. Simplify as much as possible.

a) $\dfrac{5a+15b}{4a+12b}$ b) $\dfrac{14+7x}{7x}$

c) $\dfrac{3x+3y+3z}{5x+5y+5z}$ d) $\dfrac{6x^4-12x^3}{2x^3-4x^2}$

e) $\dfrac{a+b}{a^2-b^2}$ f) $\dfrac{x^2-xy}{x^2-2xy+y^2}$

g) $\dfrac{x^2-y^2}{ax+ay}$ h) $\dfrac{12ax^2+18a^2x}{9a^2+12ax+4x^2}$

Exercise 2.20. Compute and simplify as much as possible.

a) $\dfrac{x^2}{y} \cdot \dfrac{2y^2}{3x}$ b) $\dfrac{5a^2b}{3x^2y} \cdot \dfrac{10xy^2}{5ab^2}$

c) $\dfrac{5a^2b}{x^2} : \dfrac{6ab^2}{xy}$ d) $\dfrac{7ab^2}{x^2y} : \dfrac{14ab^3}{xy^2}$

Exercise 2.21. Compute and simplify as much as possible.

a) $\dfrac{x^2-y^2}{x} \cdot \dfrac{x^2}{x+y}$ b) $\dfrac{16a^2-25b^2}{a^2-16} \cdot \dfrac{a^3-4a^2}{4a-5b}$

c) $\dfrac{x^2-2xy+y^2}{x-y} : \dfrac{3x-3y}{3}$ d) $\dfrac{x^2+6x+9}{a+b} : \dfrac{x^2+5x+6}{2ab+2b^2}$

Exercise 2.22. Simplify as much as possible.

a) $\dfrac{3x}{x^2-4} - \dfrac{6}{x^2-4}$ b) $\dfrac{x-4}{3} + \dfrac{x-3}{4} - \dfrac{x-12}{12}$

c) $\dfrac{(x+1)^2}{15} + \dfrac{(x-2)^2}{3} - \dfrac{(x-3)^2}{5}$ d) $\dfrac{3x^2-5}{4a} + \dfrac{6-2x^2}{3a}$

e) $\dfrac{x-y}{xy} - \dfrac{z-y}{yz} + \dfrac{z-x}{xz}$ f) $\dfrac{2}{x} + \dfrac{3x+1}{x^2} - \dfrac{x-2}{x^3}$

g) $\dfrac{5}{x} - \dfrac{2x-1}{x^2} + \dfrac{x+5}{x^3}$ h) $\dfrac{5}{m-n} + \dfrac{4}{m+n}$

i) $\dfrac{3a}{a^2-1} + \dfrac{4}{a+1}$ j) $\dfrac{5}{t+2} + \dfrac{2}{t-2} - \dfrac{40}{t^2-4}$

2.5 Solutions

Exercise 2.1.

a) $-2a$ b) $2xyz$
c) $4ab$ d) $5x$
e) 0 f) $-4ab$
g) $-9ab$ h) $-11a^2b+23ab^2$
i) $-11a^3x^2+a^2x^3$ j) $-3a+6c$
k) $6uv-9u^2v$ l) $4x^2y^3$

Exercise 2.2.

a) x^{12} b) $8x^6$
c) $-10x^5y^3$ d) $12a^6b^6c^6$
e) $2x^6y^9$ f) $\frac{3}{4}x^7y^6z^3$
g) $9x^4y^2$ h) $\frac{25}{49}a^4b^6c^8$
i) $128x^9y^{16}$ j) $2160x^{23}y^{27}$

Exercise 2.3.

a) $9b^3$ b) $25x^4$
c) $28x^3$ d) $37a^2b^2$

Exercise 2.4.

a) $15a+6$ b) $-5a+15$
c) $2z+z^2$ d) $2x^2-\frac{1}{2}x$
e) $-3x+6y-9z$ f) $x^2+8x+15$
g) x^2-x-2 h) $6z^2+10z-4$

Exercise 2.5.

a) $x^4-x^3+x^2-2x+7$ b) $2048y^5\cdot z^6$
c) $-2x^2+5y$ d) $-6x-12$
e) $-x+9$ f) $4x^2-13x+7$
g) $19x-y$ h) $9x-3$
i) $5y$ j) 0

Exercise 2.6.

a) $3x^3+3x^2-60x$ b) $4x^4-12x^3-8x^2+24x$
c) x^4-2x^2-x d) $12a^3-16a^2-5a+3$
e) $-x^3+9x^2-49x$ f) $9x^6-51x^5-6x^4$
g) $-11a+3b+4c$ h) x^2+2x+3

Exercise 2.7.

a) -54 b) $-\dfrac{1}{2}$

Exercise 2.8.

a) x^2+2x+1 b) x^2-6x+9
c) x^2-36 d) $x^2+10x+25$
e) $x^2-14x+49$ f) x^2-4
g) x^2-4x+4 h) x^2-49
i) $4x^2-16$ j) $16m^4-72m^2n^2+81n^4$
k) $9x^2-36$ l) $\dfrac{4}{49}x^2-\dfrac{6}{7}xy+\dfrac{9}{4}y^2$
m) $x^2y^4z^2-25$ n) $x^2-\dfrac{4}{9}$

Exercise 2.9.

a) $9x^2-24x+16=(3x-4)^2$ b) $64x^2+\dfrac{16}{3}\cdot x+\dfrac{1}{9}=\left(8x+\dfrac{1}{3}\right)^2$
c) $\left(\dfrac{1}{3}x+6\right)^2=\dfrac{1}{9}x^2+4x+36$ d) $\left(\dfrac{1}{5}x+\dfrac{3}{4}\right)^2=\dfrac{1}{25}x^2+\dfrac{3}{10}x+\dfrac{9}{16}$

Exercise 2.10.

a) x^3+3x^2+3x+1 b) $x^6-3x^4+3x^2-1$
c) $8x^3+36x^2y+54xy^2+27y^3$ d) $27x^3-216x^2+576x-512$
e) $-27x^9+108x^8-144x^7+64x^6$ f) $27x^3+54x^2y^2+36xy^4+8y^6$

Exercise 2.11.

a) $2(a+b)$ b) $s(r+4t)$
c) $3(4a+5b-3c)$ d) $2u(2u-v)$
e) $5xy(2+3y)$ f) $3b(3-10ac+c)$
g) $4a(a^3-2a^2+5a-1)$ h) $3a^2b(b-2)$
i) $3x^2y^2(y-3x)$ j) $8x^3y^2(2x^2+y)$
k) $5x^3y^2(3y^2-5x+2x^3y^2)$ l) $11r^2s^3(11rs+7s-5r^2)$

Exercise 2.12.

a) $(x-y)(n-1)$ b) $(3p-2q)(3a-10b)$
c) $r(a-2)(1+r-r^2)$ d) $(x-y)(2-x)$
e) $(4x-3)(6-5x)$ f) $17(x-2)[-1+5(x-2)]=17(x-2)(5x-11)$
g) $(5x+2y)[8-2x(2y+5x)]$ h) $(2x-1)[(2x-1)-3(x+2)+(x+4)]=(2x-1)[-3]$

Exercise 2.13.

a) $(2x+y)(a-3b)$ b) $(2a+b)(2x-3y)$
c) $3(x^2-9)(x+1)$ d) $5(x^2-4)(x+2)$
e) $(x+2)(3x+2y)$ f) $(ay+3x)(2y-x)$
g) $(1-x)(1+x^2+x^4)$ h) $(x-y)(a-b+2)$

Exercise 2.14.

a) $(x+5)^2$ b) $(x+8)^2$
c) $(x-7)^2$ d) $(2x+1)^2$
e) $(3x+2)(3x-2)$ f) $(x+5)(x-5)$

Exercise 2.15.

a) $(x+1)(x+2)$ b) $(x+2)(x+3)$
c) $(x+3)(x-1)$ d) $(x+3)(x+4)$
e) $(x+6)(x-5)$ f) $(x+7)(x+8)$
g) $(x-4)(x-5)$ h) $(x+8)(x-7)$
i) $(x+4)(x-3)$ j) $(x-6)(x-8)$

Exercise 2.16.

a) $x(x+y)^2$ b) $xy(x+y)(x-y)$
c) $x^2(x+5)(x-5)$ d) $x(2x+1)^2$
e) $x^{11}(9x-4)^2$ f) $x^{13}(4x+9)^2$
g) $(x^4+16)(x^2+4)(x+2)(x-2)$ h) $x(x^2+9)(x+3)(x-3)$
i) $2x(x-1)(x-10)$ j) $2x(2x-3)(2x+3)(4x^2+9)$

Exercise 2.17.

a) $\dfrac{30x^3y}{48x^2y^2}$ b) $\dfrac{3a^2-2a-1}{a^2-1}$
c) $\dfrac{2a^2-5a-25}{a^2-10a+25}$ d) $\dfrac{3x^2-8xy-3y^2}{9x^2-y^2}$

Exercise 2.18.

a) $\dfrac{3a}{4b^2}$ b) $-\dfrac{3a^2}{7xy}$
c) $-\dfrac{3a^3y^5}{121x^3}$ d) $\dfrac{a+b}{a-b}$

Exercise 2.19.

a) $\dfrac{5}{4}$ b) $\dfrac{2+x}{x}$
c) $\dfrac{3}{5}$ d) $3x$
e) $\dfrac{1}{a-b}$ f) $\dfrac{x}{x-y}$
g) $\dfrac{x-y}{a}$ h) $\dfrac{6ax}{2x+3a}$

Exercise 2.20.

a) $\dfrac{2xy}{3}$ b) $\dfrac{10ay}{3bx}$

c) $\dfrac{5ay}{6bx}$ d) $\dfrac{y}{2bx}$

Exercise 2.21.

a) $x(x-y)$ b) $\dfrac{a^2(4a+5b)}{a+4}$

c) 1 d) $\dfrac{2b(x+3)}{x+2}$

Exercise 2.22.

a) $\dfrac{3}{x+2}$ b) $\dfrac{6x-13}{12}$

c) $\dfrac{x^2-2}{5}$ d) $\dfrac{x^2+9}{12a}$

e) 0 f) $\dfrac{5x^2+2}{x^3}$

g) $\dfrac{3x^2+2x+5}{x^3}$ h) $\dfrac{9m+n}{(m+n)(m-n)}$

i) $\dfrac{7a-4}{a^2-1}$ j) $\dfrac{7t-46}{t^2-4}$

2.6 Chapter objectives

At the end of this chapter, the student should be able to

2.1 ☐ Compute the 4 basic operations (addition, subtraction and multiplication) on monomials and polynomials.
2.2 ☐ Raise a binomial to the square using remarkable identities.
2.3 ☐ Factorize a polynomial by factoring out a common factor.
2.4 ☐ Factorize a polynomial by grouping.
2.5 ☐ Factorize a polynomial using the remarkable identities.
2.6 ☐ Factorize a second degree trinomial.
2.7 ☐ Combine these techniques.
2.8 ☐ Expand an algebraic fraction.
2.9 ☐ Simplify an algebraic fraction.
2.10 ☐ Compute the 4 basic operations on algebraic fractions.

Chapter 3

Equations

3.1 First degree equation with one unknown

Definition. An *equation* is an equality containing one or several variables (unknowns).

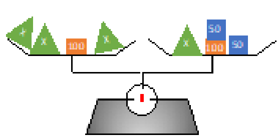
Example. $x - 5 = 7$ is an equation where $x - 5$ is called the *left side* and 7 is called the *right side*. *The unknown* in this equation is x.

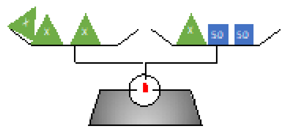
When solving an equation, the goal is to find the value(s) of the unknown such that the equality is correct. In other words, such that the left side is equal to the right side. Such a value is called *solution* of the equation. For example, 12 is the solution to the equation $x - 5 = 7$, so both the left and right side member are equal to 7. Equations whose unknown's exponent does not exceed 1 are called *first degree equations*.

How to solve first degree equations?

It is possible to perform all operations (addition/subtraction/multiplication/division) on one side as long as the same operation is also performed on the other side. The goal is to isolate the unknown x in order to find its value.

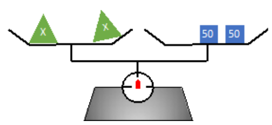
Example.

Let's solve $3x + 100 = x + 100 + 2 \cdot 50$	
	$3x + 100 = x + 100 + 2 \cdot 50.$
We remove 100 from both sides.	



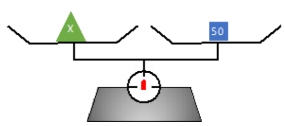
$$3\textcolor{green}{x} = \textcolor{green}{x} + 2 \cdot \textcolor{blue}{50}.$$

We remove x from both sides.



$$2\textcolor{green}{x} = 2 \cdot \textcolor{blue}{50}.$$

We divide both sides by 2.



$$\textcolor{green}{x} = \textcolor{blue}{50}.$$

The solution is $x = 50$

Example. Let's solve $3x + 5 = 17$.
We can subtract 5 from both sides of the equation: $3x + 5 - 5 = 17 - 5$.
We get $3x = 12$.
Let's divide now by 3 on both sides: $\dfrac{3x}{3} = \dfrac{12}{3}$
We therefore obtain $x = 4$.
From now on, we will present these different steps as follows:

$$\begin{array}{rcl|l} 3x + 5 & = & 17 & -5 \\ 3x & = & 12 & :3 \\ x & = & 4. & \end{array}$$

Exercise 3.1. In every cases, circle the numbers that are solutions of the equation.

a)	$3x - 15 - 4x = -9 + x - 13$	$x = 5$	$x = -\dfrac{8}{3}$	$x = \dfrac{7}{2}$	$x = 7$	$x = \dfrac{21}{6}$
b)	$4(2x + 3) = 2(4x + 6)$	$x = 0$	$x = -4$	$x = \dfrac{4}{5}$	$x = 9$	$x = 16$
c)	$x^2 - 9 = 0$	$x = 4$	$x = -3$	$x = -5$	$x = 3$	$x = 6$
d)	$\dfrac{x}{4} + \dfrac{x}{3} = x - 5$	$x = 12$	$x = -7$	$x = 6$	$x = -12$	$x = 1$
e)	$\dfrac{x+1}{2} - \dfrac{4x+3}{7} = 1$	$x = 7$	$x = 9$	$x = -4$	$x = 8$	$x = -13$
f)	$x^3 - 8 = 0$	$x = 2$	$x = 3$	$x = 8$	$x = -8$	$x = -2$

Exercise 3.2. Solve the equations.

a) $2x - 2 = -9$
b) $-3x + 4 = -1$
c) $4x - 3 = -5x + 6$
d) $5x - 4 = 2(x - 2)$
e) $4(2y + 5) = 3(5y - 2)$
f) $6(2y + 3) - 3(y + 5) = 0$

Exercise 3.3. Solve the equations.

a) $3x + 5 = 3x - 7$
b) $5x + 7 - 2x = 3x + 7$
c) $\dfrac{5}{3}x - 1 = 4 + \dfrac{2}{3}x$
d) $\dfrac{1}{5}x + 2 = 3 - \dfrac{2}{7}x$
e) $(3x - 2)^2 = (x - 5)(9x + 4)$
f) $(5x - 7)(2x + 1) - 10x(x - 4) = 0$
g) $x + \dfrac{2}{3} = \dfrac{5}{6}$
h) $\dfrac{2x - 9}{4} = 2 + \dfrac{x}{12}$

Exercise 3.4. Solve the equations.

a) $3x + 2(1 - 3x) = 5(x + 2) - 2 - x$
b) $(x - 6)^2 + (x - 4)^2 + (2x - 9)^2 = (x - 8)(6x - 8)$
c) $(5x + 1)(2x - 1) - (2x + 1)(3x - 5) = 4(x + 3)(x - 1) - 4$
d) $60 = \left(\dfrac{x - 3}{3}\right) \cdot 4$
e) $\dfrac{2x}{5} - \dfrac{3}{4} = \dfrac{7}{2}$
f) $\dfrac{x - 1}{2} + \dfrac{x + 1}{3} = 5$
g) $\dfrac{3x - 10}{6} - \dfrac{-10x - 8}{2} = \dfrac{10x - 2}{3}$
h) $\dfrac{2(5 - 2x)}{3} - \dfrac{x + 2}{4} = 4(x + 1)$
i) $\dfrac{x + 7}{5} - \dfrac{3x + 1}{6} = 3 - \dfrac{x + 7}{15}$
j) $\dfrac{2x - 4}{3} - \dfrac{x + 5}{5} - \dfrac{2x + 3}{4} = 1$

3.2 First degree equations with two unknowns

It is possible to have equations with two different unknowns, such as $x + 2y = 8$. However, an equation with two unknowns admits an infinity of solutions, for instance $(0; 4)$, $(8; 0)$, $(2; 3)$, $\left(-\frac{1}{3}; \frac{25}{6}\right)$ etc. To make the solution unique, a second equation is needed. For this reason, we will study the systems of two equations with two unknowns. To solve such a system, there are two main methods: the substitution method and the addition method.

3.2.1 Substitution method

Example.

Let's solve $\left\{\begin{array}{rcl|l} x+2y &=& 8 & \text{(I)} \\ 2x-y &=& 1 & \text{(II)} \end{array}\right.$.

$$\begin{array}{rcl|l} x+2y &=& 8 & -2y \\ x &=& 8-2y & \end{array}$$

Step 1:
In one equation, we isolate one of the unknowns, for example x. Using the equation (I), we get $x = 8-2y$

$$\begin{array}{lrcl|l} \text{(II)} & 2x-y &=& 1 & x=8-2y \\ & 2(8-2y)-y &=& 1 & \text{Distribution} \\ & 16-4y-y &=& 1 & \text{Distribution} \\ & 16-5y &=& 1 & \text{Simplification} \end{array}$$

Step 2:
In the non-used equation, we substitute (replace) the unknown obtained by the result in the first step. We then use the second equation (II) in which we replace x by $8-2y$.

$$\begin{array}{rcl|l} 16-5y &=& 1 & -16 \\ -5y &=& -15 & :(-5) \\ y &=& 3 & \end{array}$$

Step 3:
Solve the obtained equation. When solving it, we get $y = 3$.

$$\begin{array}{rcl|l} x &=& 8-2y & y=3 \\ x &=& 8-2\cdot 3 & \\ x &=& 2 & \end{array}$$

Step 4:
In the equation obtained during first step, replaces the result found in step 3.
We replace y by 3 in the equation $x = 8-2y$.

Hence, the solutions are $x = 2$ and $y = 3$,
we can write $(x; y) = (2; 3)$.

Exercise 3.5. Solve all the systems of equations using the substitution method.

a) $\left\{\begin{array}{rcl} x+2y &=& 8 \\ 3x+y &=& -1 \end{array}\right.$

b) $\left\{\begin{array}{rcl} y &=& x-1 \\ 2x &=& 2y+2 \end{array}\right.$

c) $\left\{\begin{array}{rcl} 2x+y &=& 29 \\ 3x-5y &=& 11 \end{array}\right.$

d) $\left\{\begin{array}{rcl} 2x+6y-4 &=& 0 \\ 3y-4 &=& -x \end{array}\right.$

e) $\left\{\begin{array}{rcl} 3x-y &=& -1 \\ 5x-3y &=& 1 \end{array}\right.$

f) $\left\{\begin{array}{rcl} 6x-18y &=& 90 \\ -7x+3y &=& -33 \end{array}\right.$

3.2.2 Addition method

The second method is called the addition method (or linear combinations).

Example.

Let's solve $\left\{\begin{array}{rcl|l} 2x-3 &=& -3y+6 & \text{(I)} \\ 5x-2y+5 &=& -20 & \text{(II)} \end{array}\right.$.

$$\left\{\begin{array}{rcl|l} 2x-3 &=& -3y+6 & +3y+3 \\ 5x-2y+5 &=& -20 & -5 \end{array}\right.$$

Step 1:
Put all terms containing unknowns on the left-hand side and constants on the right-hand side.

$$\left\{\begin{array}{rcl|l} 2x+3y &=& 9 & \cdot 5 \\ 5x-2y &=& -25 & \cdot(-2) \end{array}\right.$$

Step 2:
Multiply one or both equations in order to get c and $-c$ before x (or y).

$$\begin{array}{rrcl} & 10x+15y &=& 45 \\ + & -10x+4y &=& 50 \\ \hline & 19y &=& 95 \end{array}$$

Step 3:
Add both equations.

$$\begin{array}{rcl|l} 19y &=& 95 & :19 \\ y &=& 5 & \end{array}$$

Step 4:
Solve the obtained equation with one unknown.

$$\begin{array}{rcl|l} 2x+3y &=& 9 & y=5 \\ 2x+3\cdot 5 &=& 9 & -15 \\ 2x &=& -6 & :2 \\ x &=& -3 & \end{array}$$

Step 5:
Replace the solution we got in step 4 in one of the two equations from step 2 and solve.

Hence, the solutions are $x=-3$ and $y=5$,
we can write $(x;y)=(-3;5)$.

Exercise 3.6. Solve all the system of equations using the addition method.

a) $\left\{\begin{array}{rcl} 7x-y &=& 4 \\ -2x+y &=& 1 \end{array}\right.$ b) $\left\{\begin{array}{rcl} 5x+8y &=& 101 \\ 9x+2y &=& 95 \end{array}\right.$

c) $\left\{\begin{array}{rcl} 6x-3y &=& 3 \\ 2x-y &=& -1 \end{array}\right.$ d) $\left\{\begin{array}{rcl} 6x-5y &=& -15 \\ -12x+10y &=& 30 \end{array}\right.$

e) $\left\{\begin{array}{rcl} 4x-5y &=& 18 \\ 6x-8y &=& 28 \end{array}\right.$ f) $\left\{\begin{array}{rcl} 4y-3x &=& 11 \\ 5y-7x &=& 4 \end{array}\right.$

Exercise 3.7. Solve each of the systems below using the most appropriate method.

a) $\begin{cases} \dfrac{x}{2} &= \dfrac{y}{3} \\ x + \dfrac{1}{3}y &= 30 \end{cases}$

b) $\begin{cases} 3x - 2y &= 12 \\ x - 6 &= \dfrac{2y}{3} - 2 \end{cases}$

c) $\begin{cases} \dfrac{x-3y}{9} &= 5 + y \\ \dfrac{x}{3} &= 15 + \dfrac{8}{3}y \end{cases}$

d) $\begin{cases} \dfrac{6(y+2)}{5} &= \dfrac{7x}{5} - 4 \\ \dfrac{1}{3}(5x+8y) &= 24 \end{cases}$

e) $\begin{cases} \dfrac{x+2y}{2} &= \dfrac{y-1}{4} \\ \dfrac{7x+13y}{12} - y &= \dfrac{x}{2} \end{cases}$

f) $\begin{cases} \dfrac{2x}{3} - \dfrac{4}{5}y &= \dfrac{2+y}{5} \\ 2x &= 3y \end{cases}$

g) $\begin{cases} \dfrac{4x-5y}{2} + 3 &= \dfrac{2x+y}{5} \\ 8 - \dfrac{x-2y}{4} &= \dfrac{x}{2} + \dfrac{y}{3} \end{cases}$

h) $\begin{cases} \dfrac{x-3}{7} - \dfrac{2y+2}{3} &= \dfrac{y-6}{7} + 1 \\ y - \dfrac{5x+1}{3} &= 19 - 3x \end{cases}$

3.3 Second degree equations

Definition. A *quadratic equation* or *second-degree equation* is any equation where the exponent of at least one unknown is equal to two (and not more).

a) **Solving equations of the type $ax^2 + c = 0$.**

Example.

$$\begin{array}{rcl|l} x^2 - 144 &=& 0 & +144 \\ x^2 &=& 144 & \sqrt{\ } \\ x &=& \pm 12. & \end{array}$$

Remark. When using a square root or any n-th root (where n is even), the positive and negative values are always considered. Indeed, $12^2 = 144$ but also $(-12)^2 = 144$.

b) **Solving equations of the type $ax^2 + bx = 0$.**

In this situation, we can factor out x.

Example.

$$\begin{array}{rcl|l} 3x^2 - 9x &=& 0 & \text{Factor out } x \\ x(3x-9) &=& 0. & \end{array}$$

So either $x = 0$ or $3x - 9 = 0$, which means $x = 3$.

The solutions are therefore $x = 0$ and $x = 3$.

c) **Solving equations of the type $ax^2 + bx + c = 0$ using factorization.**

Example.

$$\begin{array}{rcl|l} x^2 - 5x + 6 & = & 0 & \text{Factorize} \\ (x-2)(x-3) & = & 0 & \end{array}$$

To make the product of a first number and a second number zero, the first number must be zero or the second number must be zero.

In other words, from

$$(x-2)(x-3) = 0,$$

we deduce the solutions

$$x = 2 \text{ and } x = 3.$$

d) **Solving general equations $ax^2 + bx + c = 0$.**

It is sometimes possible to factorize using remarkable identities or by trial and error. Otherwise, there is a formula for determining solutions.

$$\boxed{\text{The equation } ax^2 + bx + c = 0 \text{ admits the solutions } x_{1;2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$

This formula is called *quadratic formula*.

We define $\Delta \overset{\text{déf}}{=} b^2 - 4ac$, the *discriminant* of the equation, where Δ is the greek capital letter "*delta*".

Three possibilities may then occur:

— $\Delta < 0$ then the equation doesn't admit any (real) solution.
— $\Delta = 0$ then the equation admits an unique solution.
— $\Delta > 0$ then the equation admits two different solutions x_1 and x_2 given by

$$\boxed{x_1 = \frac{-b + \sqrt{\Delta}}{2a} \text{ and } x_2 = \frac{-b - \sqrt{\Delta}}{2a}.}$$

Example.

<table>
<tr><td colspan="2">Let's solve $x^2+6=5x$.</td></tr>
<tr><td>$$\begin{array}{rcl|l} x^2+6 &=& 5x & -5x \\ x^2-5x+6 &=& 0 & \end{array}$$</td><td>Step 1:
Move all terms to the same side of the = sign.</td></tr>
<tr><td>$a=1$, $b=-5$ and $c=6$.</td><td>Step 2:
Identify the coefficients a, b and c.</td></tr>
<tr><td>$$\begin{array}{rcl} \Delta &=& b^2-4ac \\ &=& (-5)^2-4\cdot 1\cdot 6 \\ &=& 25-24 \\ &=& 1. \end{array}$$</td><td>Step 3:
Let's compute Δ.
We notice that we will have two different solutions.</td></tr>
<tr><td>$$\begin{array}{rcl} x_{1;2} &=& \frac{-b\pm\sqrt{\Delta}}{2a} \\ &=& \frac{-(-5)\pm\sqrt{1}}{2\cdot 1} \\ &=& \frac{5\pm 1}{2} \\ && x_1=3 \text{ and } x_2=2. \end{array}$$</td><td>Step 4:
Let's use the quadratic formula.</td></tr>
<tr><td colspan="2">Hence, the solutions are $x=3$ and $x=2$.</td></tr>
</table>

Exercise 3.8. Solve.

a) $(x-2)(x-3)=0$ b) $(x+1)(x+5)=0$
c) $x(x-1)=0$ d) $(x+1)(x-2)(x-4)=0$
e) $x(2x-4)(3x-3)(5x-15)=0$ f) $(x+1)(x-1)(x+2)(x-2)(x+3)(x-3)(x+4)(x-4)=0$

Exercise 3.9. Solve.

a) $x^2=4$ b) $x^2-49=0$
c) $x^2+25=0$ d) $x^2+100=0$
e) $25x^2=9$ f) $x^2-5x=0$
g) $x^2=4x$ h) $3x^2-12=0$

Exercise 3.10. Solve using factorization.

a) $x^2-2x+1=0$ b) $x^2-10x+25=0$
c) $x^2+6x+9=0$ d) $4x^2-4x+1=0$
e) $16x^2-24x+11=2$ f) $9x^2+12x+4=0$
g) $x^2-3x-4=0$ h) $x^2+5x+6=0$
i) $x^2+2x-15=0$ j) $x^2+x-30=0$
k) $x^2-13x+42=0$ l) $x^3-9x=0$

Exercise 3.11. Solve using the quadratic formula or another method.

a) $6x^2 - 30x - 144 = 0$

b) $12x^2 + 36x - 120 = 0$

c) $16x^2 - 64x + 64 = 0$

d) $9x^2 + 42x + 69 = 0$

e) $3x^2 - 132 = 21x$

f) $4x(x + 2) = 32$

g) $2x^2 - \frac{11x}{10} - \frac{3}{10} = 0$

h) $x(x + 1) = 2(x + 1)$

i) $-\frac{x^2}{8} + (x - 1)(x + 1) + \frac{1}{8} = 0$

j) $\frac{1}{2}(9 - x) + \frac{4}{3}(x - 3)^2 - \frac{5}{6} = 0$

k) $\frac{x^2}{5} + \frac{1}{4} = \frac{x^2}{3}$

l) $(2x - 3)(3x - 2) - (3x - 1)(x + 3) = 9 - 21x$

3.4 Equations of the form $ax^4 + bx^2 + c = 0$

Since the quadratic formula only allows to solve second degree equations, it is necessary to perform a substitution of the variable.

Example.

<table>
<tr><td colspan="2">Let's solve $2x^4 - 6x^2 - 8 = 0$.</td></tr>
<tr><td>$$\begin{array}{rcl|l} 2x^4 - 6x^2 - 8 &=& 0 & u = x^2 \\ 2u^2 - 6u - 8 &=& 0 & \end{array}$$</td><td>Step 1:
Let's substitute the variable
We define u as x^2, we therefore have $x^2 = u$
and consequently $x^4 = u^2$.</td></tr>
<tr><td>$u_{1;2} = \frac{6 \pm 10}{4}$
$u_1 = 4$ and $u_2 = -1$.</td><td>Step 2:
Solve using the quadratic formula.
Watch out, we obtain the value(s) of u.</td></tr>
<tr><td>$$\begin{array}{rcl|l} x^2 &=& 4 & \sqrt{} \\ x &=& \pm 2 & \end{array}$$
$textand$
$$\begin{array}{rcl|l} x^2 &=& -1 & \sqrt{} \end{array}$$
No solution!</td><td>Step 3:
We deduce the values for x knowing that $x^2 = u$.</td></tr>
<tr><td colspan="2">Hence, the solutions are $x = 2$ and $x = -2$.</td></tr>
</table>

Exercise 3.12. Solve.

a) $x^4 - 41x^2 + 400 = 0$
b) $x^4 + 21x^2 - 100 = 0$
c) $3x^4 - 12x^2 = 0$
d) $x^4 + 9x^2 + 18 = 0$
e) $x^4 - 9x^2 + 20 = 0$
f) $8x^6 - 63x^3 - 8 = 0$
g) $x^8 - 97x^4 + 1296 = 0$
h) $x^{10} + 31x^5 - 32 = 0$

3.5 Radical equations

Definition. A *radical equation* is an equation with at least one unknown inside a radical.

Example.

Let's solve $\sqrt{2x-1} - 3 = -2x$.

$$\begin{array}{rcl|l} \sqrt{2x-1} - 3 & = & -2x & +3 \\ \sqrt{2x-1} & = & 3 - 2x & \end{array}$$

Step 1:
Isolate the radical expression.

$$\begin{array}{rcl|l} \sqrt{2x-1} & = & 3 - 2x & (...)^2 \\ \left(\sqrt{2x-1}\right)^2 & = & (3-2x)^2 & \\ 2x - 1 & = & 9 - 12x + 4x^2 & \end{array}$$

Step 2:
Square both sides
of the equal sign.
Pay attention to the remarkable identities!

$x_1 = 1$ and $x_2 = \dfrac{5}{2}$.

Step 3:
Solve the obtained equation.

$$\begin{array}{llcll} x_1 = 1 & \underbrace{\sqrt{2 \cdot 1 - 1} - 3}_{=-2} & = & \underbrace{-2 \cdot 1}_{=-2} & \text{OK} \\ x_2 = 2,5 & \underbrace{\sqrt{2 \cdot 2,5 - 1} - 3}_{=-1} & \neq & \underbrace{-2 \cdot 2,5}_{=-5} & \text{KO} \end{array}$$

Step 4:
Check the solutions!

The only solution is $x = 1$.

Exercise 3.13. Solve

a) $\sqrt{x-1} = 7$
b) $\sqrt{x+4} = -9$
c) $\sqrt{7-5x} = 8$
d) $-\dfrac{1}{2}\sqrt{2x-5} = 12$
e) $x - 2 = \sqrt{x^2 - 3x + 1}$
f) $\sqrt{2x^2 - 5x + 7} = 7 + x$
g) $x + \sqrt{3x+1} = 1$
h) $8 - 3\sqrt{2x-1} = 2$
i) $x - \sqrt{3x+25} = 15$
j) $x - \sqrt{4x-19} = 4$
k) $3(\sqrt{x} + 1) + 2\sqrt{x} = 5$
l) $3\sqrt{x-2} = 4 - \sqrt{x-2}$

3.6 Problems

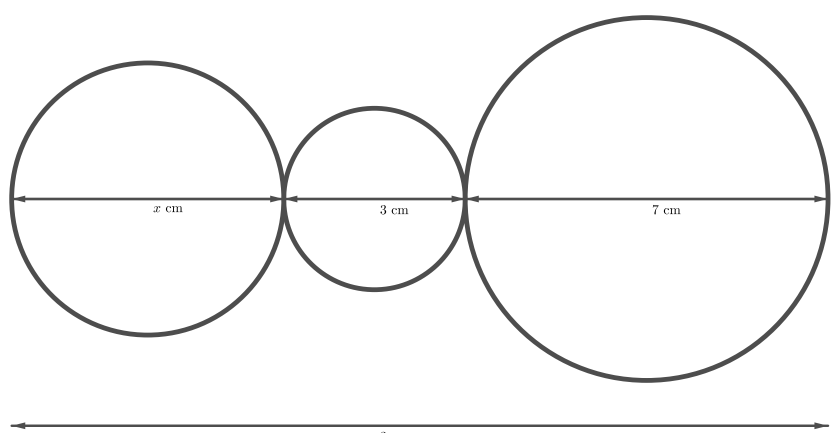
When solving mathematical problems, it is necessary to start by defining the unknowns. Very often, the final question helps us defining them correctly. Then, it is necessary to determine the equations and finally solve them.

Example.

In a store, all CDs have the same price and so do all comic books.
Marcel bought two CDs and three comic books for 53 francs. Tristan bought four CDs and one comic book for 66 francs.
Calculate the price of a CD and the price of a comic book.

Let's define $\begin{cases} x &= \text{Price of a CD} \\ y &= \text{Price of a comic book} \end{cases}$. Marcel's purchase: $2x + 3y = 53$. Tristan's purchase: $4x + 1y = 66$.	**Step 1:** Define the two unknowns.
$\begin{cases} 2x + 3y &= 53 \\ 4x + y &= 66 \end{cases}$.	**Step 2:** Convert the data of the problem into equations.
$(x; y) = (14,50; 8)$	**Step 3:** Solve the system of equations.
A CD costs 14,50 francs. A comic book costs 8 francs.	**Step 4:** Formulate the solutions in the form of a sentence.

Exercise 3.14. Using an equation, describe the situation below and solve it.



Exercise 3.15. Find 3 consecutive integers whose sum is equal to 984.

Exercise 3.16. The fuel tank of a car is filled up to one third. We add 42 liters to fill it up. What is its capacity?

Exercise 3.17. A magician asks a spectator: "think of a number, multiply it by 2, subtract 3 from the result and then multiply it by 6". The spectator announces 294. What number was he thinking of?

Exercise 3.18. A family eats in a restaurant. At the end of the meal, they give a 50 franc note to pay the bill. The waiter gives back the change, which is 8,80 francs. Knowing that the price of the meal is 10,30 francs per person, how many people compose this family?

Exercise 3.19. If we double the difference between 24 and the quadruple of the number sought, we get 15 plus the double of that number.

Exercise 3.20. In a hotel, half of the guests are Belgian, one third Dutch, one seventh French and the last three people are Spanish. How many guests are there in the hotel?

Exercise 3.21. A rectangle has a length of $5x$ and a width of $4x$. If you increase its length by 18 cm and double its width, this rectangle becomes a square. What are the initial dimensions of the rectangle?

Exercise 3.22. A father is 45 years old and his son 10 years old. How long before the father is twice as old as his son?

Exercise 3.23. If we add 12 to a number and then multiply that sum by 5 and subtract 72 from that product and divide that difference by 4 we get the number itself. What is this number?

Exercise 3.24. A staircase has 22 steps. If each step was heightened by 1,6 cm, 2 steps could be taken off. What is the height of the staircase?

Exercise 3.25. A book has 240 pages with the same number of lines on each page. Putting 3 more lines per page, would result in a book with 24 fewer pages. What is the total number of lines in this book?

Exercise 3.26. If one side of a square is increased by 1,30 m and the other side is decreased by 80 cm, the resulting rectangle will have the same area as the original square. What's the side of this square?

Exercise 3.27. We shared 710 francs among 40 people. Each man received 15 francs and each woman 20 francs. How many men and women were there?

Exercise 3.28. On a farm, there are chickens and rabbits. We count 70 legs and 25 heads. How many chickens and rabbits are there?

Exercise 3.29. How many 5 franc coins and 2 franc coins do you need to obtain the sum of 115 francs, knowing that you need a total of 32 coins?

Exercise 3.30. A herd is composed of camels and dromedaries. There are 180 heads and 304 humps. How many animals of each species are there?

Exercise 3.31. 36 people ate in a restaurant. The adult menu was 22 francs and the children's menu was 9 francs. Knowing that the owner's revenue was 623 francs, how many children's and adult menus were served?

Exercise 3.32. The revenue from one match amounts to 36'500 francs. The spectators can choose between two possibilities. Either take a seat in the stands at 50 francs or take a seat in the popular area at 30 francs. There were 1'000 spectators. How many spectators took a seat in the stands?

Exercise 3.33. For 5 meters of silk and 4 meters of cloth we paid 256 francs and for 4 meters of silk and 5 meters of cloth we paid 248 francs. What is the price per meter of each type of fabric?

Exercise 3.34. The factory A has twice as many workers as the factory B. One quarter of the workers from A and one fifth of the workers from B fill seven 25-seat buses. How many workers are there in each factory?

Exercise 3.35. A computer engineer charges 120 francs per hour when he works and 80 francs per hour when he sends his employee. The customer receives a bill of 1'120 francs. Given that the engineer spent half as much time at the customer's premises as his employee, how much time did each spend with the customer?

Exercise 3.36. Ten years ago, a father's age was six times his son's. Ten years from now, it will be twice as old. What are the current ages?

Exercise 3.37. A father is now four times older than his son. In 6 years, he will be 3 years younger than the triple of his son's age. How old are they?

Exercise 3.38. An amount was shared equally among a certain number of people. If there had been 6 more people, each would have received 2 francs less. If there had been 3 fewer people, each would have received 2 francs more. Determine the number of people and the amount they each received.

Exercise 3.39. All the classes in a school have the same number of students. As a result of a fire, 6 of the classrooms were no longer usable. We then had to add 5 students per class. After the visit of the fire expert, 10 more classes were also closed for safety reasons. As a result, 15 more students had to be put back in each of the classrooms that remained in good condition. How many students does the school have?

Exercise 3.40. Find two consecutive integers such that the sum of their squares equals 545.

Exercise 3.41. Find two real numbers whose sum is 10 and whose product is 24.

Exercise 3.42. At noon, the hands of a clock are 17 cm apart and 85 cm at nine o'clock. What are the lengths of the two hands?

Exercise 3.43. A rectangular lawn has a length which is twice its width. A 3 meters wide pathway surrounds the lawn. Calculate the width of the lawn, knowing that the total area, lawn and pathway, is 360m^2.

Exercise 3.44. A salesman buys a series of reproductions for a total of 672 francs. If each reproduction had cost four francs less, he could have bought three more. How many reproductions did he buy and at what price?

Exercise 3.45. A postman start talking to one person. He asks:

— "Do you have daughters?"

— "Yes, three."

— "How old are they?"

— "Well, it's simple: the product of their age is 36 and the sum of their ages equals the house number across the street."

— "I need one more hint."

— "Yes, I'm sorry. My eldest daughter is blonde."

— "Oh yes, now I know!"

How old are the three girls?

3.7 Solutions

Exercise 3.1.

a)	$x = 5$	$x = -\frac{8}{3}$	$\boxed{x = \frac{7}{2}}$	$x = 7$	$\boxed{x = \frac{21}{6}}$
b)	$\boxed{x = 0}$	$\boxed{x = -4}$	$\boxed{x = \frac{4}{5}}$	$\boxed{x = 9}$	$\boxed{x = 16}$
c)	$x = 4$	$\boxed{x = -3}$	$x = -5$	$\boxed{x = 3}$	$x = 6$
d)	$\boxed{x = 12}$	$x = -7$	$x = 6$	$x = -12$	$x = 1$
e)	$x = 7$	$x = 9$	$x = -4$	$x = 8$	$\boxed{x = -13}$
f)	$\boxed{x = 2}$	$x = 3$	$x = 8$	$x = -8$	$x = -2$

Exercise 3.2.

a) $x = -\frac{7}{2}$ b) $x = \frac{5}{3}$

c) $x = 1$ d) $x = 0$

e) $y = \frac{26}{7}$ f) $y = -\frac{1}{3}$

Exercise 3.3.

a) No solution b) Infinity of solutions

c) $x = 5$ d) $x = \frac{35}{17}$

e) $x = -\frac{24}{29}$ f) $x = \frac{7}{31}$

g) $x = \frac{1}{6}$ h) $x = \frac{51}{5}$

Exercise 3.4.

a) $x = -\frac{6}{7}$ b) No solution

c) $x = 5$ d) $x = 48$

e) $x = \frac{85}{8}$ f) $x = \frac{31}{5}$

g) $x = -\frac{18}{13}$ h) $x = -\frac{14}{67}$

i) $x = -\frac{39}{7}$ j) $x = -\frac{245}{2}$

Exercise 3.5.

a) $(x; y) = (-2; 5)$ b) Infinity of solutions

c) $(x; y) = (12; 5)$ d) No solution

e) $(x; y) = (-1; -2)$ f) $(x; y) = (3; -4)$

Exercise 3.6.

a) $(x; y) = (1; 3)$
b) $(x; y) = (9; 7)$
c) No solution
d) Infinity of solutions
e) $(x; y) = (2; -2)$
f) $(x; y) = (3; 5)$

Exercise 3.7.

a) $(x; y) = (20; 30)$
b) Infinity of solutions
c) $(x; y) = (45; 0)$
d) $(x; y) = (8; 4)$
e) $(x; y) = (1; -1)$
f) No solution
g) $(x; y) = \left(\frac{2652}{211}; \frac{1806}{211}\right)$
h) $(x; y) = \left(\frac{152}{11}; \frac{10}{11}\right)$

Exercise 3.8.

a) $x = 2$ and $x = 3$
b) $x = -1$ and $x = -5$
c) $x = 0$ and $x = 1$
d) $x = -1$, $x = 2$ and $x = 4$
e) $x = 0$, $x = 2$, $x = 1$ and $x = 3$
f) $x = 1$, $x = -1$, $x = 2$, $x = -2$, $x = 3$, $x = -3$, $x = 4$ and $x = -4$

Exercise 3.9.

a) $x = 2$ and $x = -2$
b) $x = 7$ and $= -7$
c) No solution
d) No solution
e) $x = \frac{3}{5}$ and $x = -\frac{3}{5}$
f) $x = 0$ and $x = 5$
g) $x = 0$ and $x = 4$
h) $x = 2$ and $x = -2$

Exercise 3.10.

a) $x = 1$
b) $x = 5$
c) $x = -3$
d) $x = \frac{1}{2}$
e) $x = \frac{3}{4}$
f) $x = -\frac{2}{3}$
g) $x = 4$ and $x = -1$
h) $x = -2$ and $x = -3$
i) $x = -5$ and $x = 3$
j) $x = -6$ and $x = 5$
k) $x = 6$ and $x = 7$
l) $x = 0, x = 3$ and $x = -3$

Exercise 3.11.

a) $x = -3$ and $x = 8$
b) $x = -5$ and $x = 2$
c) $x = 2$
d) No solution
e) $x = -4$ and $x = 11$
f) $x = -4$ and $x = 2$
g) $x = -\frac{1}{5}$ and $x = \frac{3}{4}$
h) $x = 2$ and $x = -1$
i) $x = 1$ and $x = -1$
j) No solution
k) $x = \sqrt{\frac{15}{8}}$ and $x = -\sqrt{\frac{15}{8}}$
l) $x = 0$

Exercise 3.12.

a) $x = 4$, $= -4$, $x = 5$ and $x = -5$
b) $x = 2$ and $x = -2$
c) $x = 0$, $x = 2$ and $x = -2$
d) No solution
e) $x = \sqrt{5}$, $x = -\sqrt{5}$, $x = 2$ and $x = -2$
f) $x = -\frac{1}{2}$ and $x = 2$
g) $x = 2$, $x = -2$, $x = 3$ and $x = -3$
h) $x = -2$ and $x = 1$

Exercise 3.13.

a) $x = 50$
b) No solution
c) $x = -\frac{57}{5}$
d) No solution
e) $x = 3$
f) $x = 21$ and $x = -2$
g) $x = 0$
h) $x = \frac{5}{2}$
i) $x = 25$
j) $x = 5$ and $x = 7$
k) $x = \frac{4}{25}$
l) $x = 3$

Exercise 3.14. $3x = x + 10$, so $x = 5$ cm.

Exercise 3.15. 327, 328 and 329.

Exercise 3.16. 63 liters.

Exercise 3.17. He was thinking of 26.

Exercise 3.18. 4 people.

Exercise 3.19. $3, 3$.

Exercise 3.20. 126 guests.

Exercise 3.21. Length: 30 cm, width: 24 cm.

Exercise 3.22. The father will be twice as old as his son in 25 years.

Exercise 3.23. 12.

Exercise 3.24. $3, 52$ m.

Exercise 3.25. 6'480 lines.

Exercise 3.26. Side of the square: $2, 08$ m.

Exercise 3.27. 18 men and 22 women.

Exercise 3.28. 15 chickens and 10 rabbits.

Exercise 3.29. 15 "2 francs coins" and 17 "5 francs coins".

Exercise 3.30. 124 camels and 56 dromedaries.

Exercise 3.31. 13 children's menus and 23 adult menus.

Exercise 3.32. 325 spectators

Exercise 3.33. Silk: 32 francs per meter and 24 francs per meter of cloth.

Exercise 3.34. 500 workers for the factory A and 250 workers for the factory B.

Exercise 3.35. 4 hours for the engineer and 8 hours for his employee.

Exercise 3.36. The father is 40 years old ans his son 15 years old.

Exercise 3.37. 9 and 36 years old.

Exercise 3.38. 12 people receive 6 francs

Exercise 3.39. 36 classes of 25 students each, which means 900 students in total.

Exercise 3.40. 16 and 17 or -17 and -16.

Exercise 3.41. 6 and 4.

Exercise 3.42. 51 cm and 68 cm.

Exercise 3.43. 9 m.

Exercise 3.44. He bought 21 reproductions which cost 32 francs per item.

Exercise 3.45. 2 years old, 2 years old and 9 years old.

3.8 Chapter objectives

At the end of this chapter, the student should be able to

3.1 ☐ Solve a first degree equation with one unknown.
3.2 ☐ Solve a system of two equations with two unknowns using the substitution method.
3.3 ☐ Solve a system of two equations with two unknowns using the addition method.
3.4 ☐ Solve a system of two equations with two unknowns using the most appropriate method.
3.5 ☐ Solve a second degree equation using the quadratic formula or another method.
3.6 ☐ Solve an equation of the form $ax^4 + bx^2 + c = 0$.
3.7 ☐ Solve an equation with radicals.
3.8 ☐ Solve a problem by converting data into equation(s).

Chapter 4

Functions

4.1 Introduction

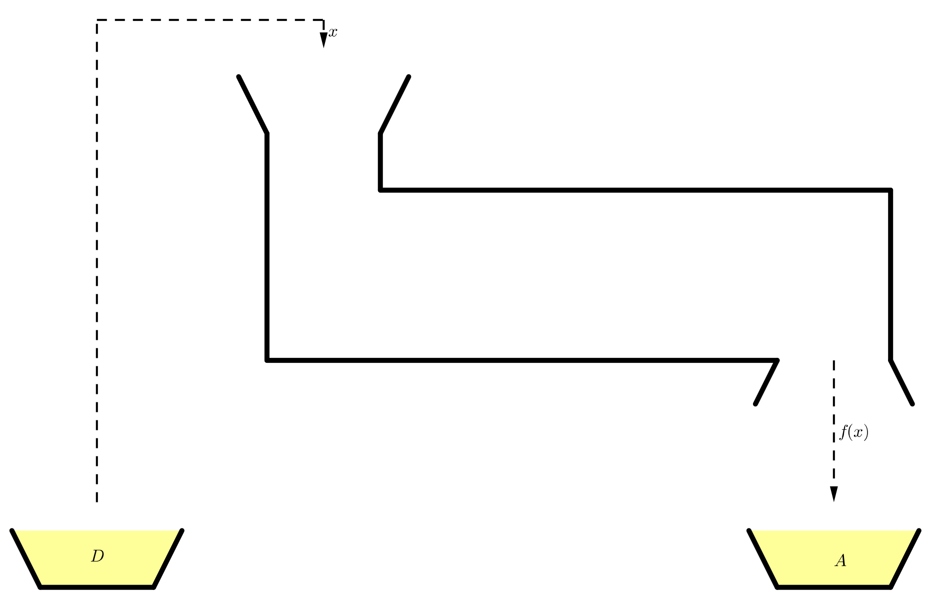
The term *function* is used in everyday language. For example, the price of a train ticket depends on the length of the trip. They say the price is a *function* of the length of the trip. The area of a disc depends on its radius. Its area is therefore a *function* of its radius.

4.2 Notion of function

Definition. We call *function* any relation that associate to every number $x \in D$ (domain) one and only one number $y = f(x) \in A$ (codomain).

We often denote a function as follows:

$$\begin{array}{rcccl} f & : & D & \to & A \\ & & x & \mapsto & f(x) \end{array} .$$



1. The element $y = f(x)$ is called *image* of x under f.
2. The element x is called *preimage* of y under f.
3. A formula for calculating the images is called *functional notation* of f.

Example. With 1 Swiss franc, you get $0,87$ euro. The number of euros you get will depend on the number of francs you change. We then say that the number of euros is a function of the number of Swiss francs.

It is possible to represent this function in different ways:

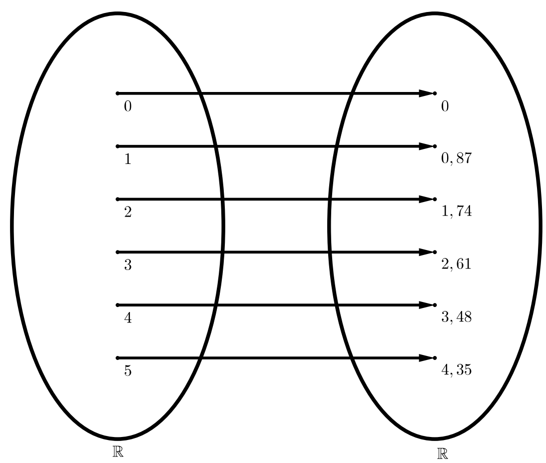
Table of values

The inconvenience of a table of values is that it does not make it possible to know what the values of f are besides the ones that appear in it.

x	0	1	2	3	4	5
$f(x)$	0	$0,87$	$1,74$	$2,61$	$3,48$	$4,35$

Mapping diagram

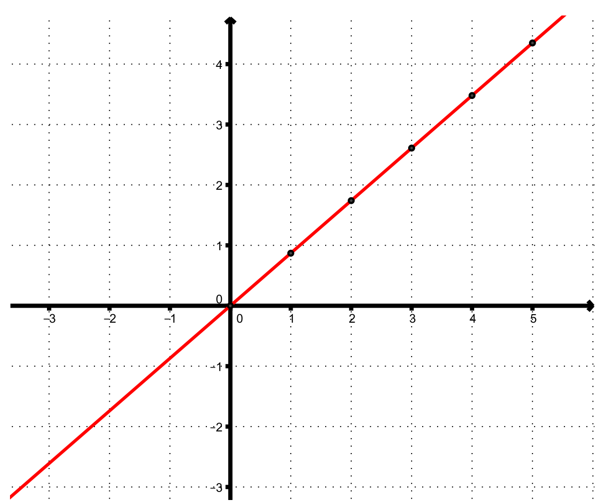
The problem with the mapping diagram is the same as the one with a table of values. However, it has the advantage of emphasizing the domain and the codomain and is more coherent than a table of values when the domain and the codomain are not numerical.



Graph

To sketch a graph, you just have to choose a value of x, taken from the domain, and calculate its image $f(x)$. x will be the first coordinate (*the abscissa*) of the point and $y = f(x)$ will be the second (*the ordinate*). In other words, the point on the vertical corresponding to x will be at $y = f(x)$. After calculating the coordinates of several points, simply connect the points by hand.

Very easy to use and relatively accurate, the graphical representation of a function is however restricted to one region. Here, for example, the graph does not show how the function behaves when $x < -3$ (x smaller than -3) and when $x > 5$ (x greater than 5).



Verbal form

"To a certain number, we associate its product with $0,87$".

Depending on the complexity of the function, it can be tedious to understand the sentence describing it.

Functional notation

The functional notation of f is denoted by

$$\begin{array}{rcccl} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & 0,87x \end{array}$$

or

$$f(x) = 0,87x$$

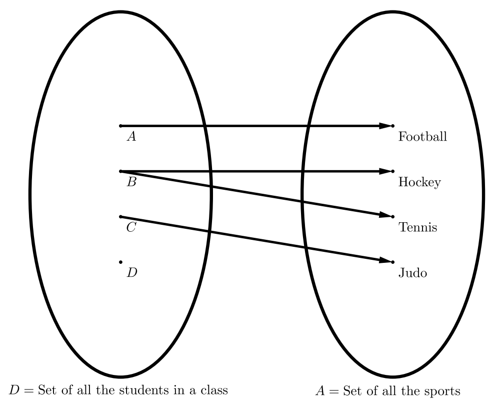
or

$$y = 0,87x.$$

The functional notation is the best way to describe a function, because by knowing it, one can construct an table of values, a mapping diagram and a graph, while the opposite is not always possible.

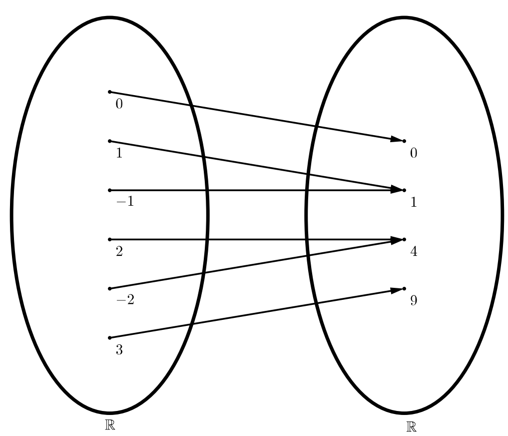
Example.

If $D =$ Set of all the students in a class and $A =$ Set of all the sports, the relation, which associates each student with the sport(s) he or she enjoys, is not a function, because two sports are associated with B and none with D.



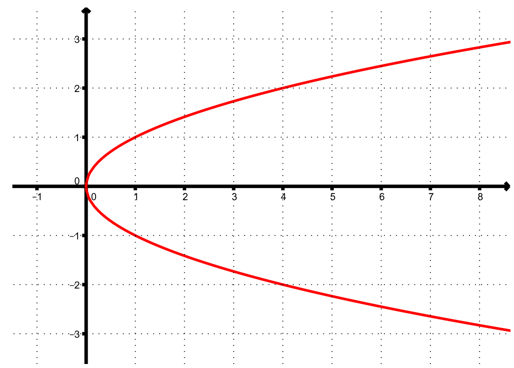
Example.

The correspondence here, which to each real number x associates its square, is a function, because any real number admits one and only one square.



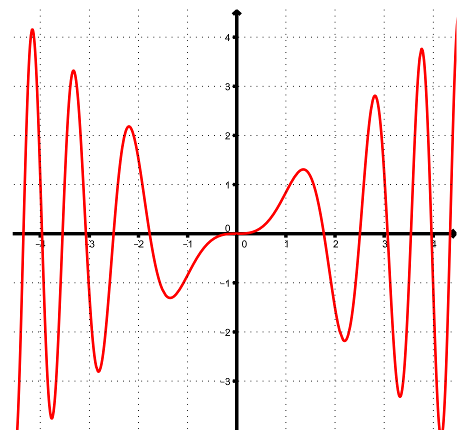
Example.

This curve does not represent a function, because two images are associated with $x = 2$ (among others).



Example.

This curve is a function, because each $x \in D = \mathbb{R}$ admits one and only one image.



Exercise 4.1. Let f be a function defined by

$$\begin{array}{rccl} f & : & \mathbb{Z} & \rightarrow & \mathbb{R} \\ & & x & \mapsto & x^2+5 \end{array}.$$

a) What is the domain and the codomain of f?

b) What is the functional notation of f?

c) What is the image of 3 under f?

d) What is the image of -2 under f?

Exercise 4.2. Fill in the following tables of values representing functions.

a)

3	1	10			x
-9	-3	-30	48	-66	

b)

-10	0	1	5	10	x
95	-5	-4			

c)

4		15	11		x
$1,\overline{3}$	6	5	$3,\overline{6}$	$-1,\overline{6}$	

d)

100	200	300		60	x
20	30	40	55		

e)

9	36		100		x
$1,5$	3	4	5	$4,5$	

f)

	4	1	5	6	x
8	64	1	125		

g)

-1	0	6	$\frac{3}{2}$		x
-5	-3		0	13	

h)

-20	20	0	-25	25	x
15	25	5	20	30	

Exercise 4.3. Let f be a function defined by

$$f(x) = \frac{x-1}{x^2-4}.$$

Compute

a) $f(-3)$ b) $f(0)$
c) $f(1)$ d) $f(2)$

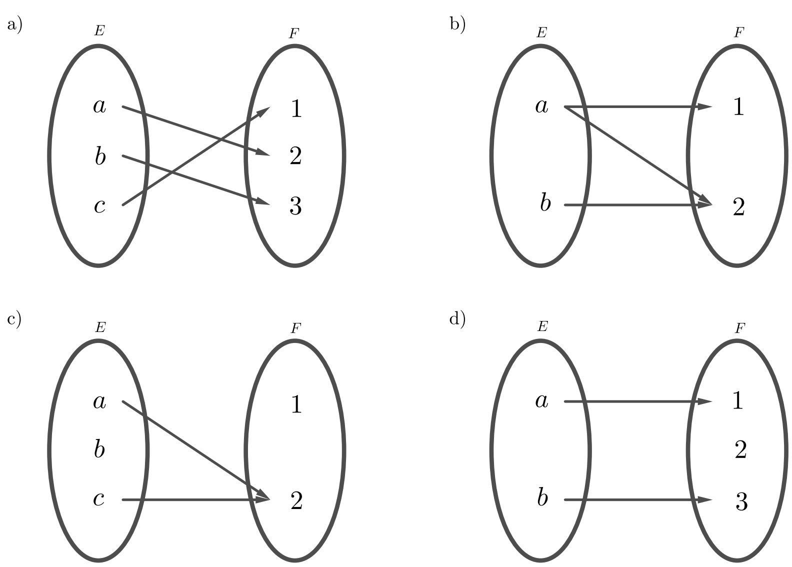
Exercise 4.4. Let f be a function defined by $f(x) = x^2 - 2x + 3$. Determine

a) $f(3)$ b) $f(-1)$
c) $f(a)$ d) $f(3a)$
e) $f(k-1)$ f) $f(2k-3)$
g) $f(2k) - 3$ h) $f(2+k) - f(2)$

Exercise 4.5. Graphically represent the following functions.

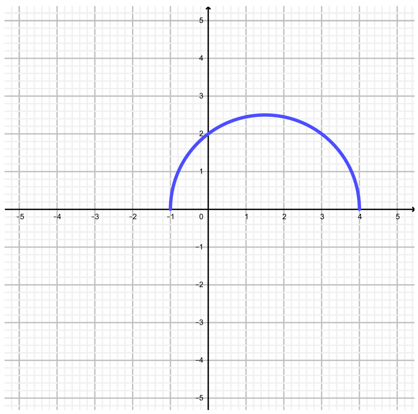
a) $f(x) = -x + 4$ b) $f(x) = -3x$
c) $f(x) = x$ d) $f(x) = -x$
e) $f(x) = -3$ f) $f(x) = |x|$
g) $f(x) = x^2$ h) $f(x) = x^3$
i) $f(x) = x^2 - 1$ j) $f(x) = \sqrt{x}$
k) $f(x) = -x^2 + 4$ l) $f(x) = |x^2 - 4|$

Exercise 4.6. Which of the following diagrams correspond to a function from E to F?

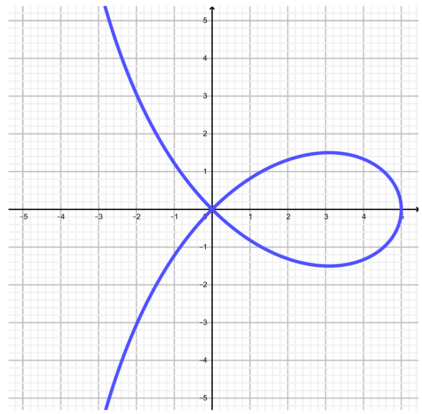


Exercise 4.7. Using the vertical line test, determine whether the following graphs represent a function or not.

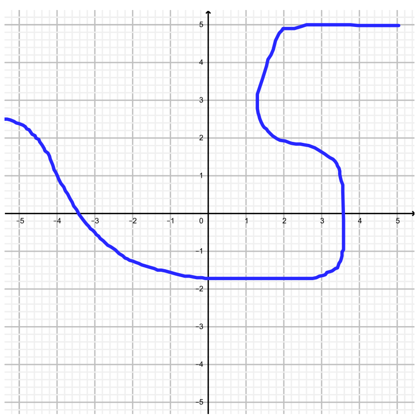
a)



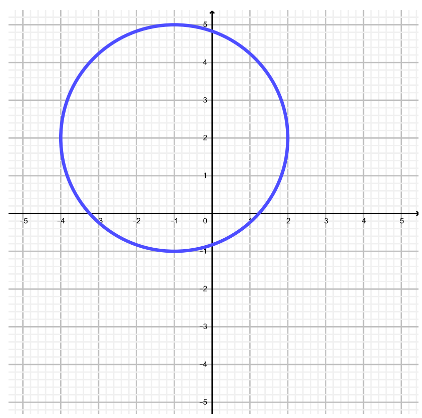
b)



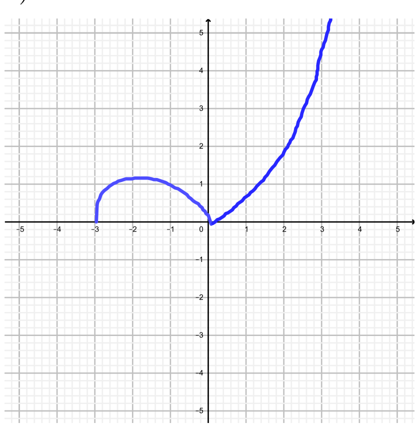
c)



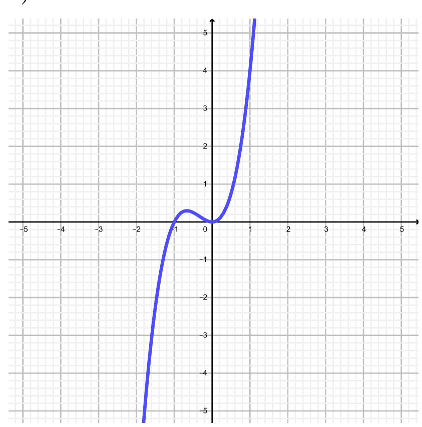
d)



e)



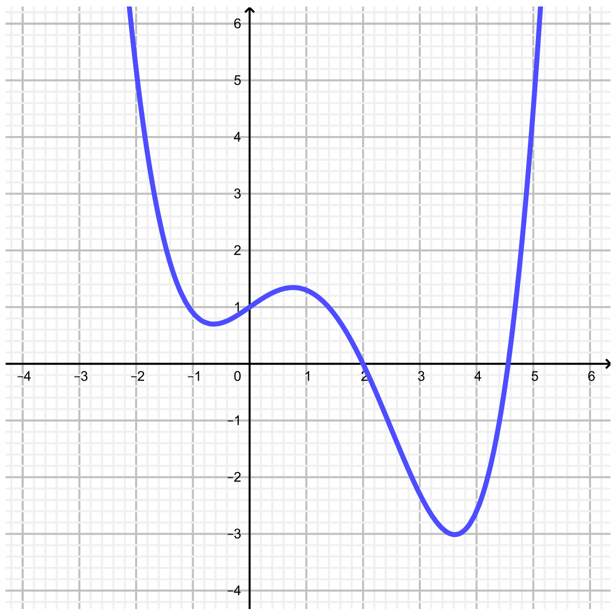
f)



Exercise 4.8. Explain why we can't find a function that at x associates y.

x	-2	1	0	2	-1	1
y	-4	3	-3	5	2	4

Exercise 4.9. The figure below is a graph of a function f.



By graphical reading, estimate

a) The value of $f(0)$.

b) The value of $f(-2)$.

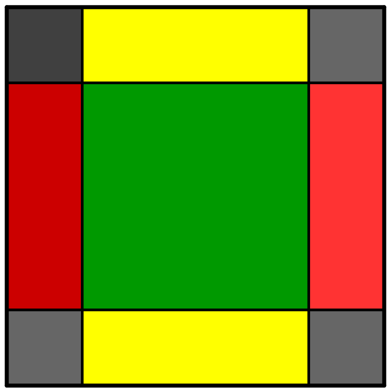
c) The values of x knowing that $f(x) = 0$.

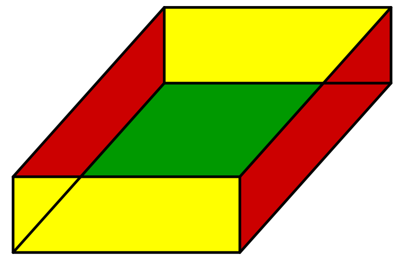
d) The values of x knowing that $f(x) = 1$.

e) The value of a knowing that the equation $f(x) = a$ admits only one solution? What's this solution?

f) The values of x knowing that $f(x) = x$.

Exercise 4.10. A company manufactures boxes without lids by cutting four identical squares whose side's length is x from the four corners of a metal plate with the dimensions 10 cm $\times$ 10 cm, and then folding up the edges.





a) Compute the volume of the box with $x = 3$.

b) Determine the algebraic expression that expresses the volume of the box $V(x)$ as a function of the length x.

4.3 Solutions

Exercise 4.1.

a) $D = \mathbb{Z}$ and $A = \mathbb{R}$

b) $f(x) = x^2 + 5$

c) $f(3) = 14$

d) $f(-2) = 9$

Exercise 4.2.

a)

3	1	10	-16	22	x
-9	-3	-30	48	-66	$-3x$

b)

-10	0	1	5	10	x
95	-5	-4	20	95	$x^2 - 5$

c)

4	18	15	11	-5	x
$1,\overline{3}$	6	5	$3,\overline{6}$	$-1,\overline{6}$	$\frac{x}{3}$

d)

100	200	300	450	60	x
20	30	40	55	16	$\frac{x}{10} + 10$

e)

9	36	64	100	81	x
$1,5$	3	4	5	$4,5$	$\frac{\sqrt{x}}{2}$

f)

2	4	1	5	6	x
8	64	1	125	216	x^3

g)

-1	0	6	$\frac{3}{2}$	8	x
-5	-3	9	0	13	$2x - 3$

h)

-20	20	0	-25	25	x
15	25	5	20	30	$\|x + 5\|$

Exercise 4.3.

a) $f(-3) = -\dfrac{4}{5}$

b) $f(0) = \dfrac{1}{4}$

c) $f(1) = 0$

d) $f(2)$ is not defined

Exercise 4.4.

a) 6

b) 6

c) $a^2 - 2a + 3$

d) $9a^2 - 6a + 3$

e) $k^2 - 4k + 6$

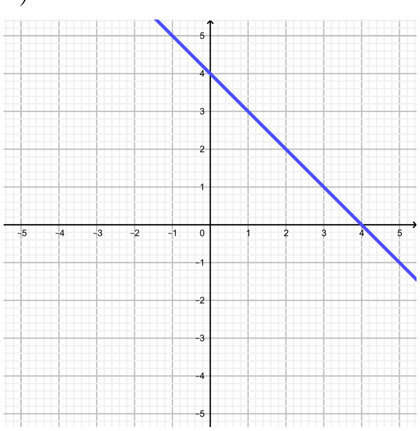
f) $4k^2 - 16k + 18$

g) $4k^2 - 4k$

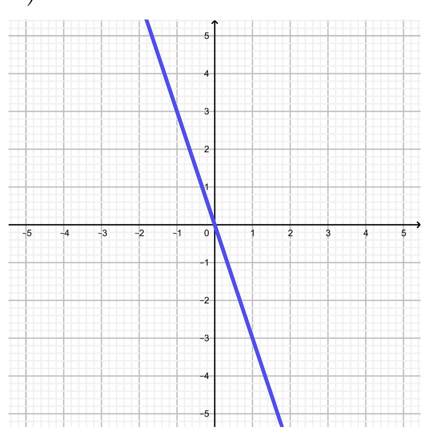
h) $k^2 + 2k$

Exercise 4.5.

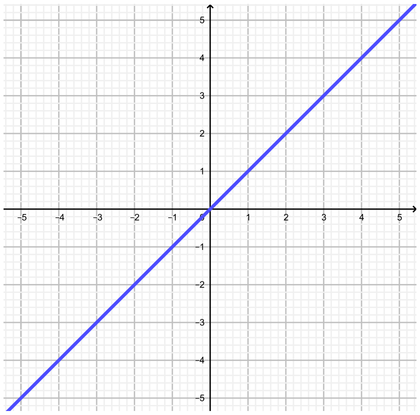
a)



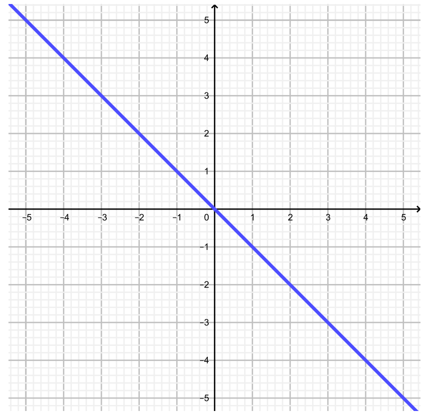
b)



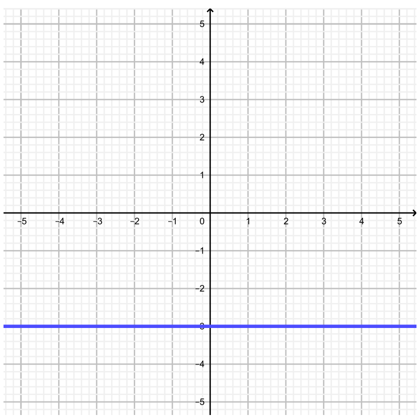
c)



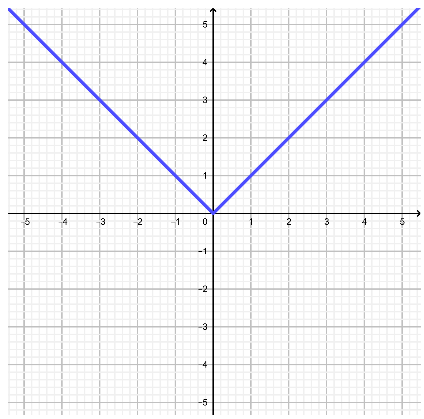
d)



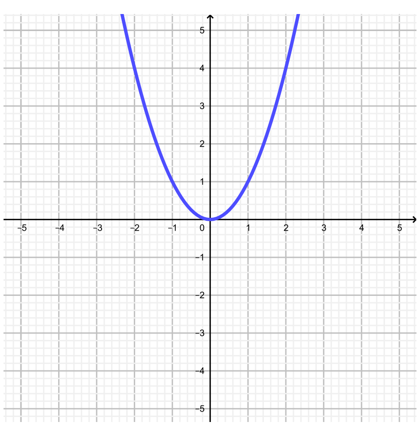
e)



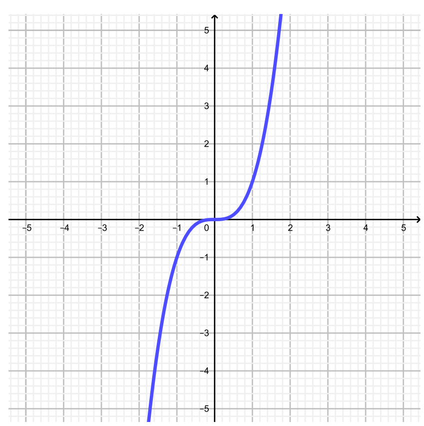
f)



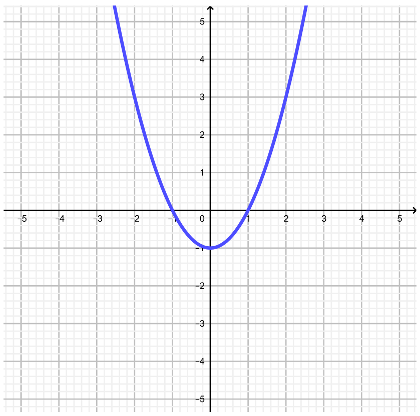
g)



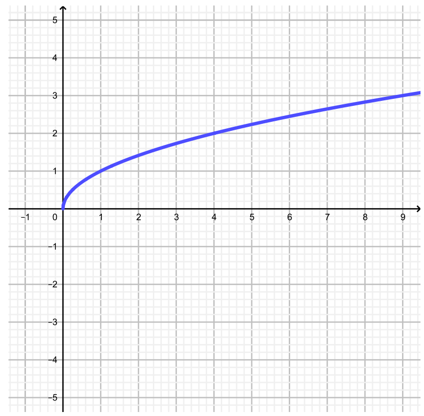
h)



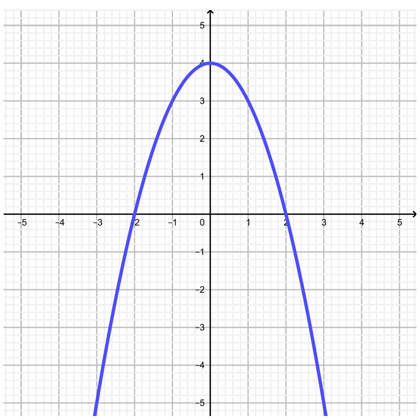
i)



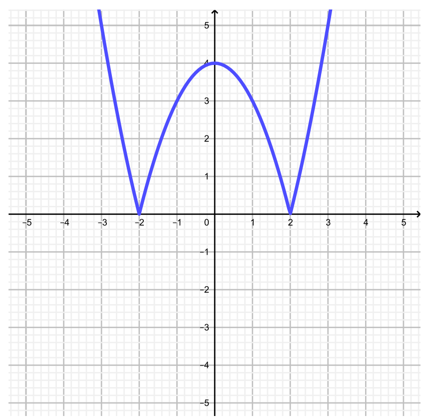
j)



k)



l)



Exercise 4.6. a) and d).

Exercise 4.7.

a) Yes	b) No
c) No	d) No
e) Yes	f) Yes

Exercise 4.8. 1 has two different images.

Exercise 4.9.

a) $f(0) = 1$	b) $f(-2) \cong 5$
c) $x = 2$ and $x \cong 4,56$	d) $x \cong -1, x = 0, x \cong 1,4$ and $x \cong 4,75$
e) $a = -3$ and $x \cong 3,62$	f) $x \cong 1,19$ and $x = 5$

Exercise 4.10.

a) $V = 48$ cm^3.

b) $V(x) = x \cdot (10 - 2x)^2$.

4.4 Chapter objectives

At the end of this chapter, the student should be able to

4.1 ☐ Know the vocabulary of the functions.
4.2 ☐ Evaluate a function for different values of x.
4.3 ☐ Sketch the graph of a function.
4.4 ☐ Read the graph of a function.
4.5 ☐ Determine whether a relation is a function or not.

Chapter 5

First degree functions

Definition. A function is said to be a *first degree function* if the degree of its functional notation is 1. We can distinguish two types:

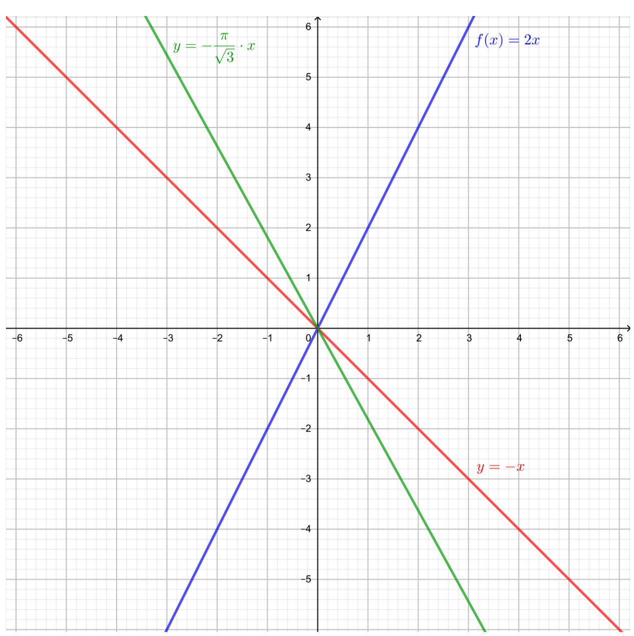
5.1 Without y-intercept

Definition. A function without any y-intercept has only a factor between x and $f(x)$. Its notation is of the form

$$\boxed{f(x) = m \cdot x \text{ or } y = m \cdot x \text{ with } m \in \mathbb{R}.}$$

Example. $f(x) = 2x$, $y = -x$, $y = -\dfrac{\pi}{\sqrt{3}} \cdot x$ are first degree function without any y-intercept.

Graphically, such a function is a straight line passing through the origin. The value of m determines the *slope* of this line.



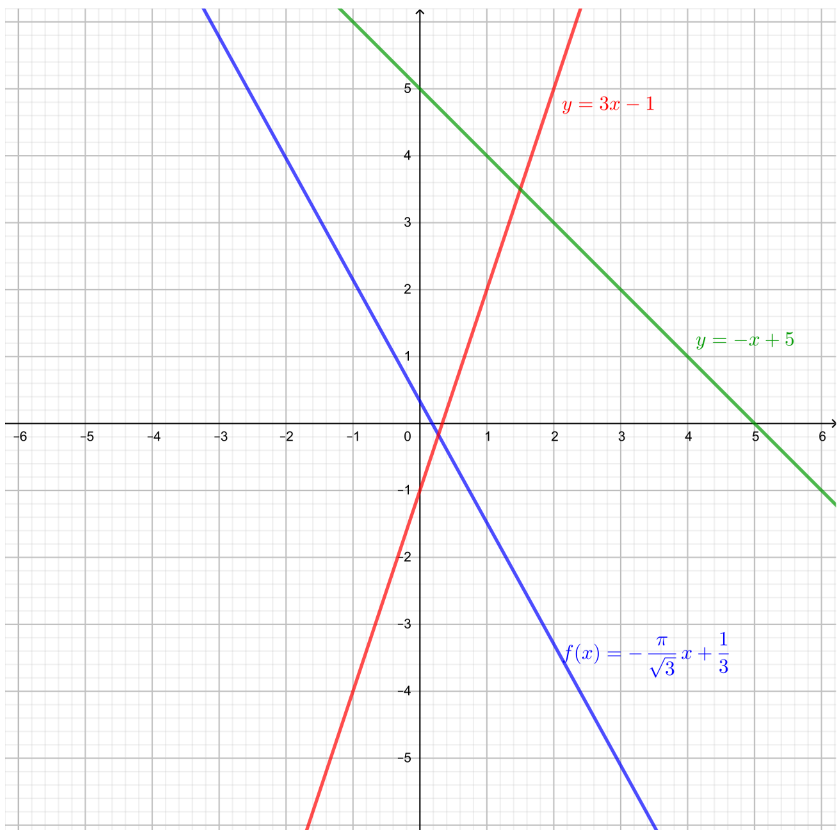
5.2 With y-intercept

Definition. A function with a y-intercept has a functional notation of the form

$$\boxed{f(x) = mx + h \text{ or } y = mx + h \text{ with } m, h \in \mathbb{R}.}$$

Example. $f(x) = 3x - 1$, $y = -x + 5$, $y = -\frac{\pi}{\sqrt{3}}x + \frac{1}{3}$ are first degree function with y-intercept.

Graphically, such a function is a straight line which doesn't pass through the origin. The value of m determine the *slope* and h is *the y-intercept* (which means the height where the line crosses the y-axis).

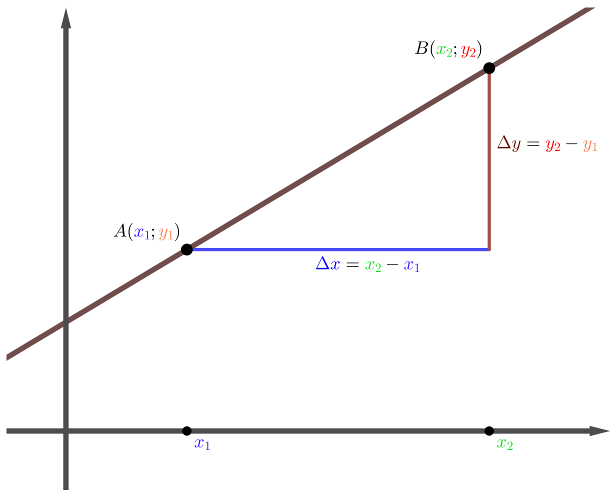


Exercise 5.1. Sketch the following functions.

a) $y = 2x$	b) $y = -2x$
c) $y = \frac{1}{2}x$	d) $y = -1$
e) $y = x + 3$	f) $y = 2x + 1$
g) $y = -2x - 3$	h) $y = 4x - 6$
i) $x + y = 0$	j) $2(x - y) = 6$

5.3 Slope of a line

Definition. The *slope* of a line is the quotient of the vertical difference by the horizontal difference between two points of the line.



Theorem. *If a line is passing through the points* $(x_1; y_1)$ *and* $(x_2; y_2)$*, then the slope is*

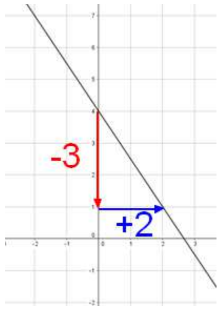
$$\boxed{m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.}$$

Remark.

The slope of a line can also be expressed in percent, for example on road signs. If the slope is 10%, this means that over a horizontal distance of 100 m, one goes up or down by a height of 10 m.

Example.

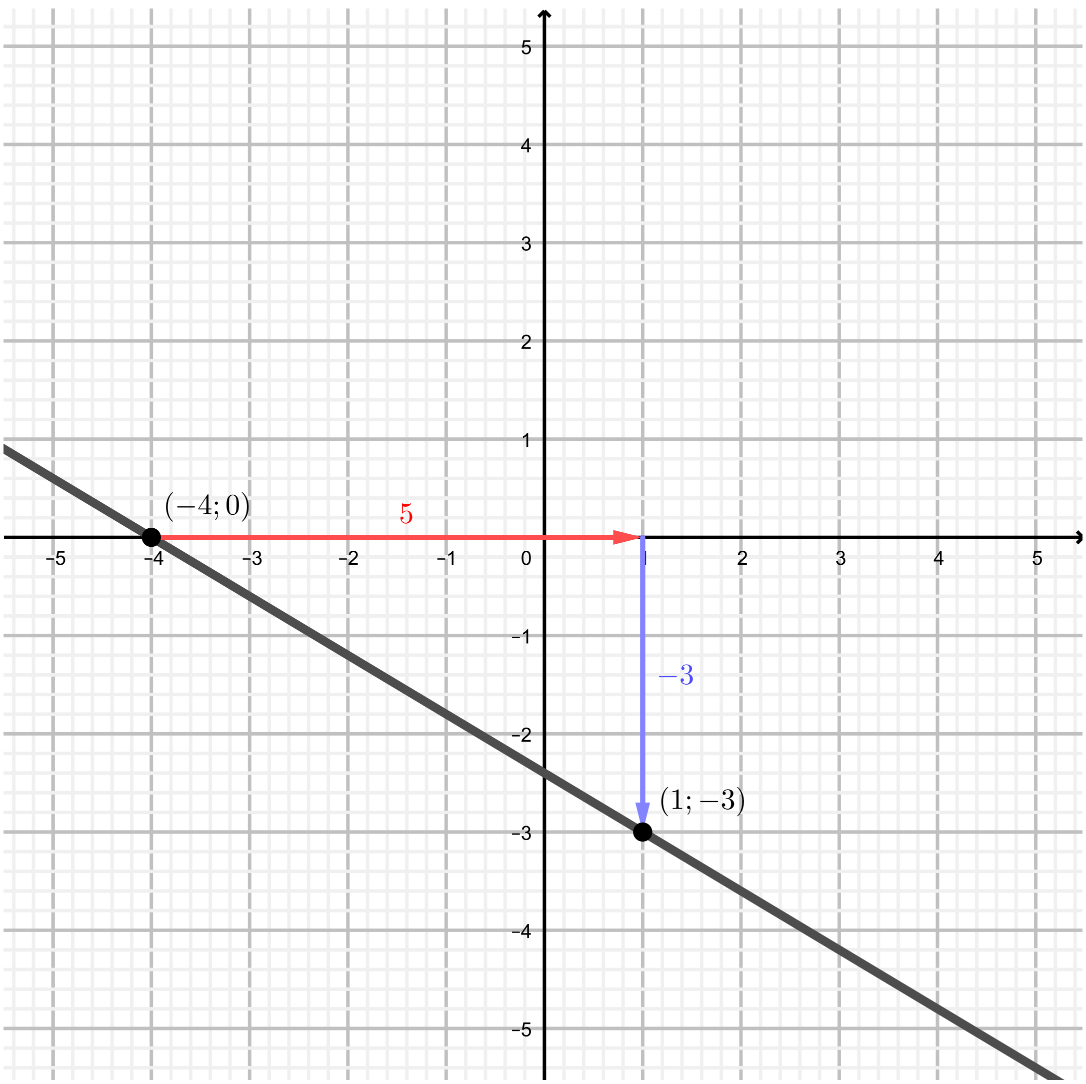
1. **How can we find the slope by graphical reading?**



Choose two points on the line and calculate the vertical difference and the horizontal difference between both points. In this example, $\Delta y = -3$ and $\Delta x = 2$. The slope is therefore $m = -\frac{3}{2}$.

2. **How do we find the slope of a line from two of its points?**
Let $(1;-3)$ and $(-4;0)$, be two points of a line. Its slope will be

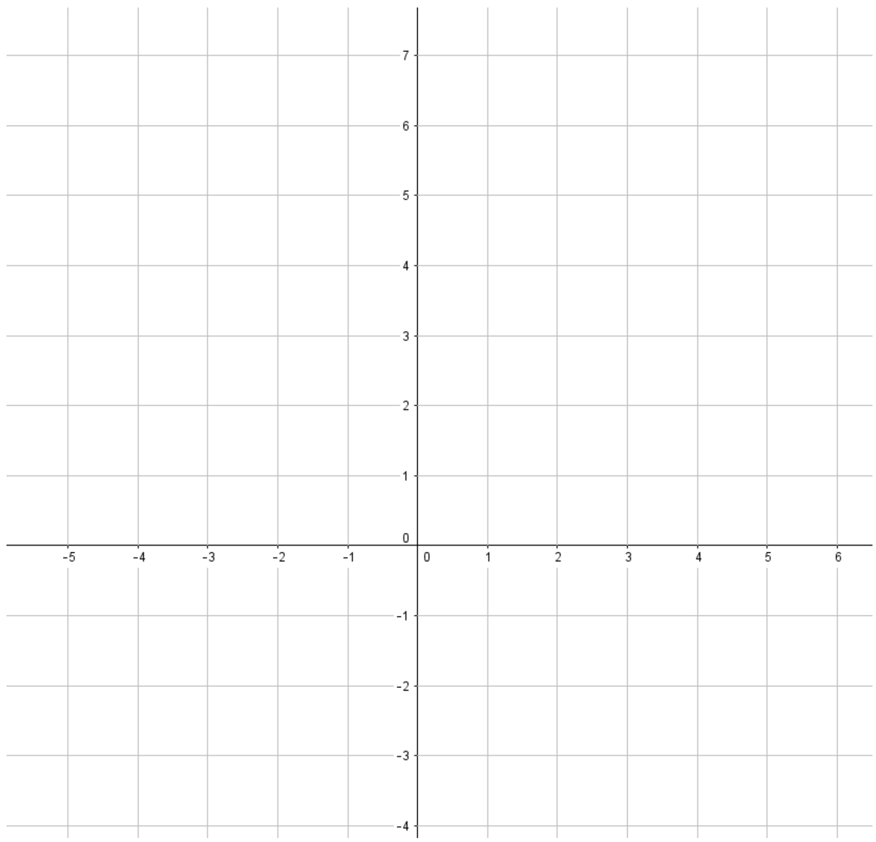
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-3)}{-4-1} = \frac{3}{-5} = -\frac{3}{5}.$$



5.4 Graph

Before drawing a line from its equation, first make sure that the equation of the line is expressed in the slope intercept form $y = mx + h$. Then, we identify the y-intercept (which gives us a first point) and afterwards we use the slope to find a second point. Finally we just have to connect them to get the line. An alternative way is to find two points using a table of values.

Example. Sketch the lines of equation $y = -\dfrac{5}{3}x + 2$ and $y - 2x = 0$.



Exercise 5.2. Determine the slope of the following lines and sketch their graph.

a) $y = x + 3$	b) $y = 2x - 2$
c) $y = 3x + 2$	d) $y = \dfrac{1}{2}x + 2$
e) $y = -x + 4$	f) $y = -2x - 3$
g) $y = -3x - 1$	h) $x + 2y = 4$

5.5 Equation of a line

If we have the graph of a first degree function, we just have to "count the little squares" to find the slope m and look where the line intersects the y-axis to get the value of the y-intercept h.

What if you don't have an accurate graph? Or even worse, if you don't have any graphs at all? To find the equation of a line, knowing two points is sufficient. Using them, we can first determine the slope $m = \frac{\Delta y}{\Delta x}$ (see section 5.3). Finally, by reusing the basic equation of a first degree function $y = mx + h$, we can replace the value of m by the value just found before and x, y by the coordinates of a known point. We then obtain an equation with only one unknown left that we just have to solve to find the value of h.

Example. Find the equation of the line passing through the points $(-2; 2)$ and $(8; -1)$.

First, it's possible to compute the slope of this line.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-2}{8-(-2)} = -\frac{3}{10}.$$

We know that a equation of a line is of the form

$$y = mx + h.$$

As $m = -\frac{3}{10}$, the equation can be written

$$y = -\frac{3}{10}x + h.$$

As the line goes through the point $(8; -1)$, it means that if $x = 8$, then $y = -1$. We therefore can replace x and y by those values in order to find h.

$$\begin{array}{rcl|l} -1 & = & -\frac{3}{10} \cdot 8 + h & \text{Computations} \\ -1 & = & -\frac{24}{10} + h & \\ -1 & = & -\frac{12}{5} + h & \cdot 5 \\ -5 & = & -12 + 5h & +12 \\ 7 & = & 5h & :5 \\ h & = & \frac{7}{5}. & \end{array}$$

So, the equation of the line is

$$y = -\frac{3}{10}x + \frac{7}{5}.$$

Remark. The same result would have been obtained if the point $(-2; 2)$ was used instead of $(8; -1)$. The important thing is to use a point from the line. The equation of this line could also have been obtained by solving the system of equations

$$\left\{ \begin{array}{rcl} m \cdot (-2) + h & = & 2 \\ m \cdot 8 + h & = & -1 \end{array} \right. .$$

Exercise 5.3. Sketch the graph of f such that

a) $f(-1) = 2$ and the slope of the graph of f is -2.

b) $f(0) = -1$ and the slope of the graph of f is $\frac{3}{2}$.

c) $f(2) = 0$ and the slope of the graph of f is $-\frac{3}{5}$.

d) $f(4) = 5$ and the slope of the graph of f is 0.

Exercise 5.4. Determine the functional notation of the function whose graph has a slope of -3 and passes through the point $A(4; 7)$.

Exercise 5.5. Determine the functional notation of the function whose graph is a line going through points A and B when

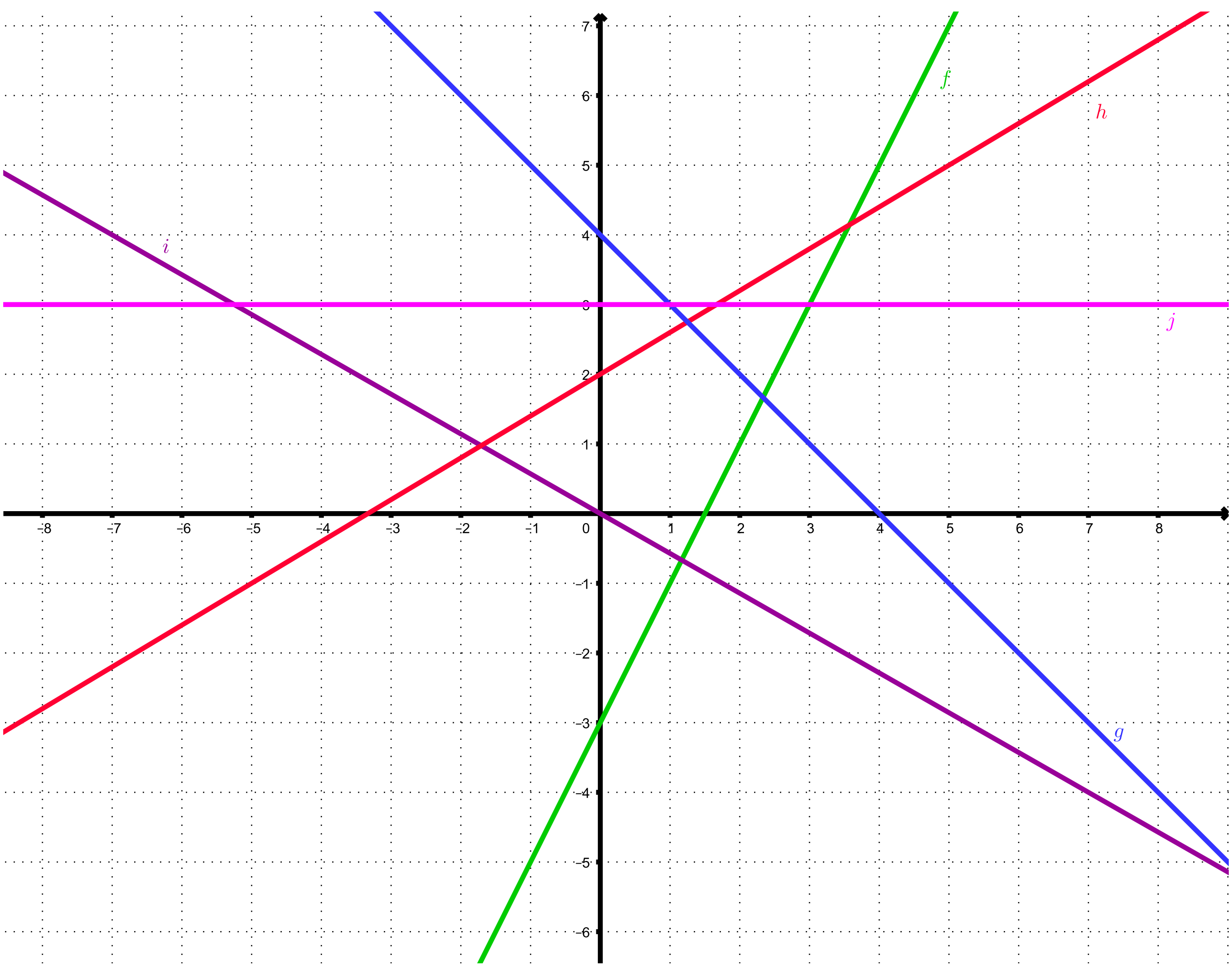
a) $A(1; 5)$ and $B(6; 20)$

b) $A(-1; -9)$ and $B(3; 11)$

c) $A(-8; -12)$ and $B(4; -3)$

d) $A(-4; 1)$ and $B(10; -7)$

Exercise 5.6. Find the functional expression of the five functions whose graphs are the lines below.



Exercise 5.7. Find the abscissa of the point $P(x; 13)$ knowing that the points P, $Q(-1; 7)$ and $R(3; -1)$ are aligned.

5.6 Special lines

Some lines are characterized by special properties.

1. **Horizontal lines**
   These are lines with a slope of 0. Their equations are of the form $y = 0 \cdot x + h$, therefore, $y = h$.

   Example.

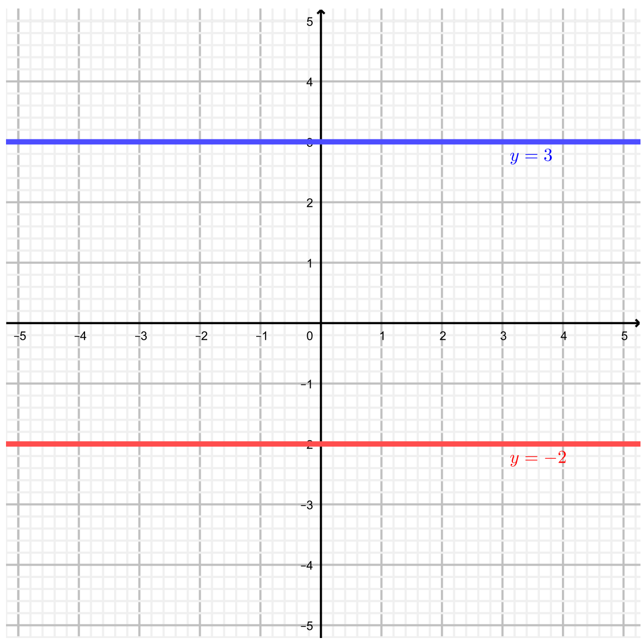


Figure 5.1: Graphs of $y = 3$ and $y = -2$.

2. **Vertical lines**
   These are lines with an infinite slope. All points on such a line have the same abscissa. The equation of such a line is of the form $x = n$.

Example.

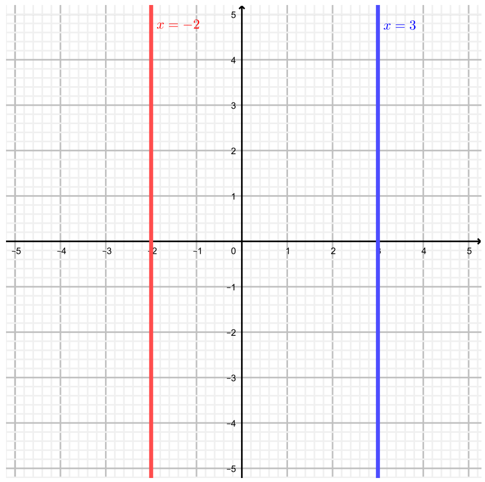
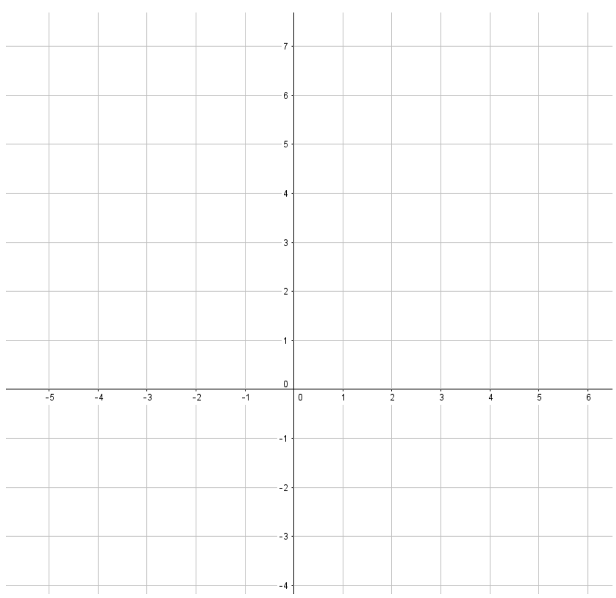


Figure 5.2: Graphs of $x = 3$ and $x = -2$.

3. **Parallel lines**
   Two straight lines are parallel if and only if they have exactly the same slope.

   Example. Are those two lines parallel?

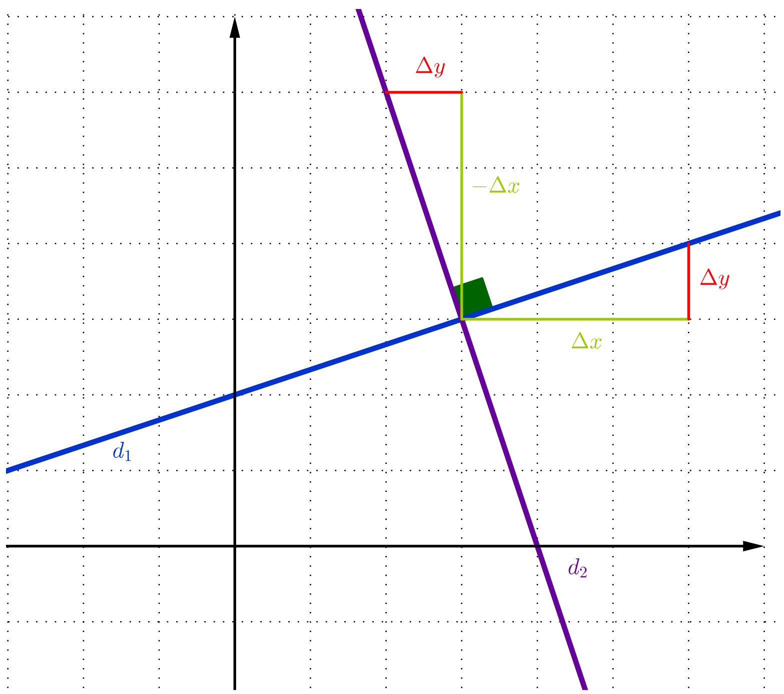
$$y = 3x + 2 \text{ and } 9x - 3y = -12.$$



4. **Perpendicular lines**
   Two lines are perpendicular if and only if the product of their slope is -1.

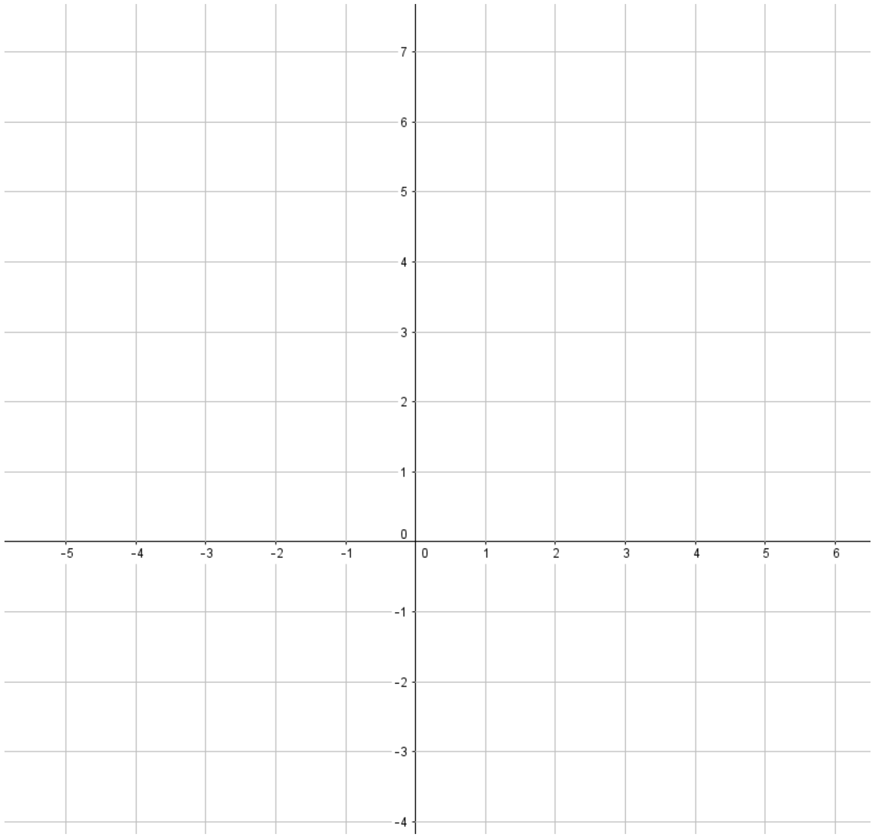
   Proof. d_1 has a slope of
   $$m_1 = \frac{\Delta y}{\Delta x}.$$
   As d_2 is perpendicular to d_1, we obtain d_2 by performing a $90°$ counterclockwise rotation of d_1.

   Then, Δx becomes a negative vertical increase and Δy becomes a positive horizontal increase.



   d_2 has then a slope of
   $$m_2 = \frac{-\Delta x}{\Delta y}.$$
   As a result,
   $$m_1 \cdot m_2 = \frac{\Delta y}{\Delta x} \cdot \frac{-\Delta x}{\Delta y} = -1.$$
   □

   Example. Are those two lines perpendicular?
   $$2x + 3y = 1 \text{ and } 6x - 4y - 1 = 0.$$



Remark. If a line has a slope of m and we want to find the slope of a perpendicular line, we can apply the method of "*the opposite reciprocal*". We thereby obtain $-\dfrac{1}{m}$ for the slope of the perpendicular line we are looking for.

Exercise 5.8. Sketch the following lines $d_1 : x = 4$ and $d_2 : y = -2$.

Exercise 5.9. Determine the equation of

a) The line parallel to $y = -3x + 2$ and passing through the origin.

b) The line parallel to $y = \dfrac{3x+5}{2}$ and passing through the point $A(6; 1)$.

c) The line perpendicular to $y = -3x + 2$ and passing through the origin.

d) The line perpendicular to $y = 2x$ and passing through the point $B(4; 2)$.

Exercise 5.10. Find the equation of the line passing through the point $P(3; -5)$ and parallel to the line of equation $2x + 2y = 4$.

Exercise 5.11. Find the equation of the line passing through the point $P(3; -5)$ and perpendicular to the line of equation $3x + 2y = 6$.

5.7 Intersection of two lines

To illustrate how useful intersections of lines can be, let's start with a real-life example.

Example. There are different types of offers for the journey from La Chaux-de-Fonds to Neuchâtel by train. We will select three of them:

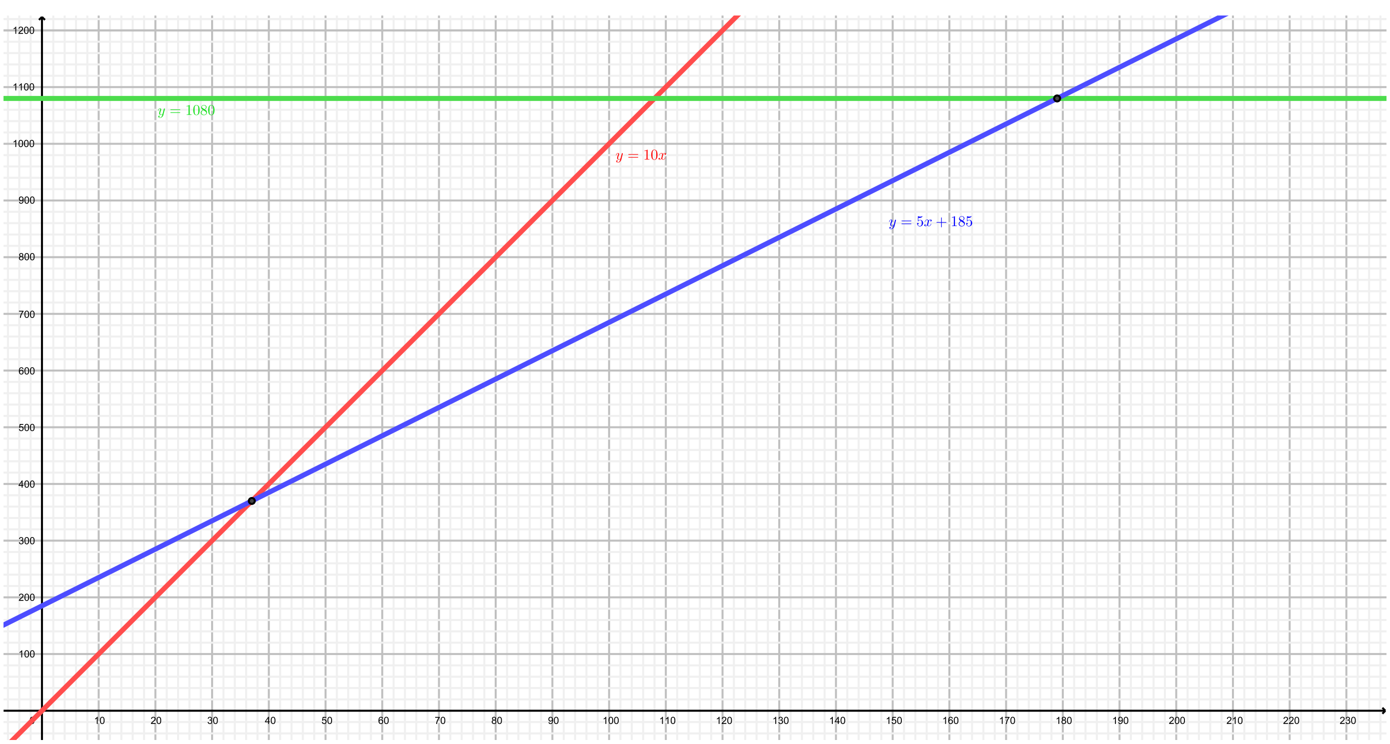
1. Full price each time: 10 francs per ride.
2. "Demi-tarif" : 185 francs travelcard and 5 francs per ride.
3. Annual pass : 1'080 francs.

Therefore, it is interesting to choose the cheapest option. In other words, from how many rides per year should we take one option rather than another one? To answer this question, let's first turn these options into functional notations.

Let's define
$x =$ "number of rides" and $y =$ "money spent".
These three types of options can be expressed using the following functional notations.

1. $y = 10x$
2. $y = 5x + 185$
3. $y = 1'080$

Here are their graphical representations:



The intersections of these lines represent the exact moment when one offer will become cheaper than another. Therefore, if we can determine the intersection of two lines, we will know the number of rides for which one offer is cheaper than the other.

Let's algebraically determine the coordinates of the intersection points. The point of intersection of two lines is the point whose coordinates satisfy the two line equations involved. In other words, "the coordinates where the lines are identical". Let's start by calculating the point of intersection between the full price and "Demi-tarif".

This involves solving the system of equations with two unknowns:

$$\begin{cases} y &= 10x \\ y &= 5x + 185 \end{cases}.$$

If we solve it using the substitution method or the addition method, we get:

$$(x; y) = (37; 370).$$

This means that after 37 trips, the "Demi-tarif" travelcard costs less than option 1 (full price). For exactly 37 trips, regardless of whether option 1 or 2 is used, we will have paid exactly 370 francs. Similarly, we can calculate the intersection point of the "Demi-tarif" travelcard and the annual pass. We get the point $(179; 1'080)$. So from 179 rides per year, it is better to have the annual pass.

Exercise 5.12. Determine the coordinates of the intersection point of the f and g in each of the following situations.

a) $f(x) = -x + 5$ and $g(x) = 3x + 1$

b) $f(x) = 3x - 6$ and $g(x) = -2x + 4$

c) $f(x) = 2x - 4$ and $g(x) = -x + 5$

d) $f(x) = 5x - 2$ and $g(x) = 3x + 4$

e) $f(x) = -2x + 8$ and $g(x) = 12$

f) $f(x) = -7x$ and $g(x) = -3x$

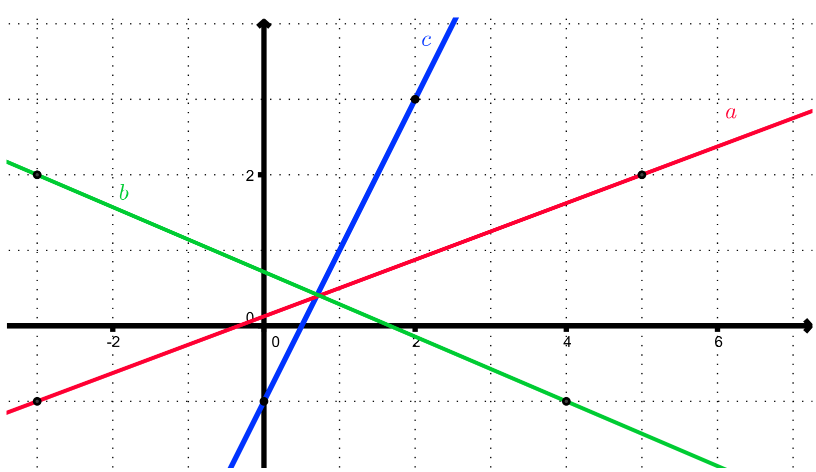
g) $f(x) = -2x + 1$ and $g(x) = -2x - 3$

h) $f(x) = 3x + 4$ and $g(x) = 3x + 4$

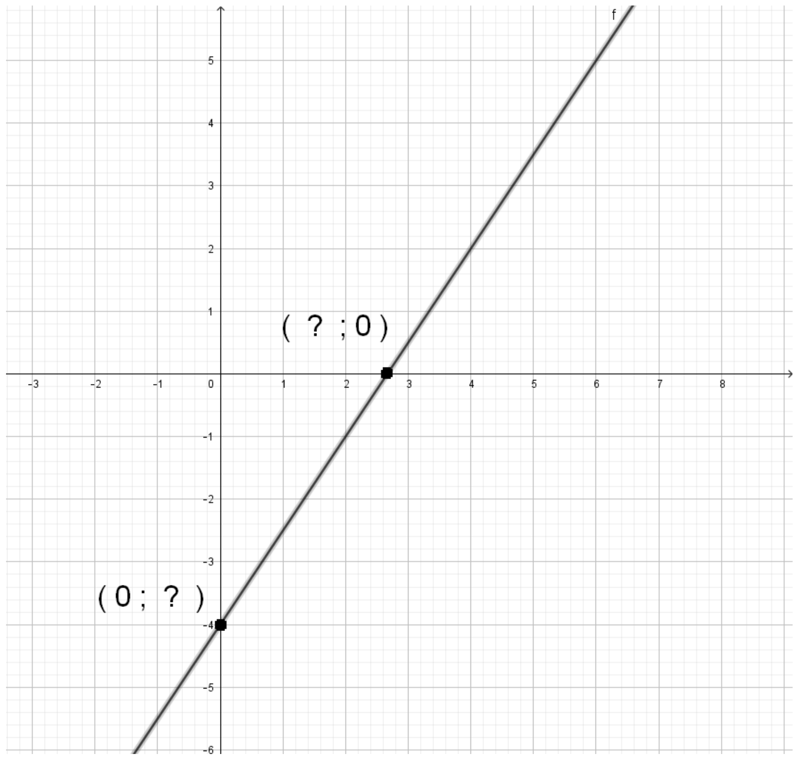
Exercise 5.13. Find algebraically the coordinates of the point of intersection between the lines d_1 and d_2.

$$d_1 : -3x + 2y = 6 \text{ and } d_2 : 2x - 8y - 16 = 0.$$

Exercise 5.14. Do the lines a, b and c intersect at the same point or do they form a triangle?



5.8 Intersection of a line and the axes



1. **Intersection with the x-axis**

   When the line crosses the x-axis, the y-coordinate of the intersection point of the line and the x-axis is 0. Therefore, to determine algebraically the coordinates of this point of intersection, we put $y = 0$ and solve the equation to find x.

   Example. Let $y = \frac{3}{2}x - 4$ be an equation of a line.

   We put $y = 0$.

   We get:

   $$\begin{aligned} 0 &= \tfrac{3}{2}x - 4 \\ x &= \tfrac{8}{3}. \end{aligned}$$

   The intersection point is therefore $\left(\frac{8}{3}; 0\right)$.

2. **Intersection with the y-axis**

   Following a similar thinking, we just have to put $x = 0$ and find the value of y.

   Example. Let $y = \frac{3}{2}x - 4$ be an equation of a line. We put $x = 0$.

   We get:

   $$y = \frac{3}{2} \cdot 0 - 4 = -4.$$

   The intersection point is therefore $(0; -4)$.

Exercise 5.15. Determine the coordinates of the intersection point of the following lines with the axes.

a) $y = 2x - 4$ b) $3x + 5y = 6$

5.9 Applications

Exercise 5.16. A sports club offers two options.

Option A:
The fan pays 6,50 francs each time he attends a match.
Option B:
The fan pays a membership fee of 28 francs and 3 francs each time he attends a match.

a) Complete the table below.

Number of matches	4	6		
Price paid with option A			65	
Price paid with option B				73

b) Express, for each option, the price paid by the fan as a function of the number of matches x he wants to attend.

c) Represent in a graph in the same coordinate system, the lines that define each option.

d) a) Determine graphically for how many matches is the price paid the same regardless of the option chosen. Give this price.
   b) Verify by showing your calculations, the results found in part a.

Exercise 5.17. We can obtain the temperature in degrees Celsius from the Fahrenheit value by the function f defined by

$$f(x) = \frac{5}{9}x - \frac{160}{9}.$$

.

a) What temperature in degrees Celsius corresponds to a temperature of 36 Fahrenheit?

b) At what temperature do you read the same value on both scales?

c) At what temperature is the Fahrenheit value twice as high as the Celsius value?

Exercise 5.18. In Switzerland, life expectancy at birth for a woman can be modelled by the formula

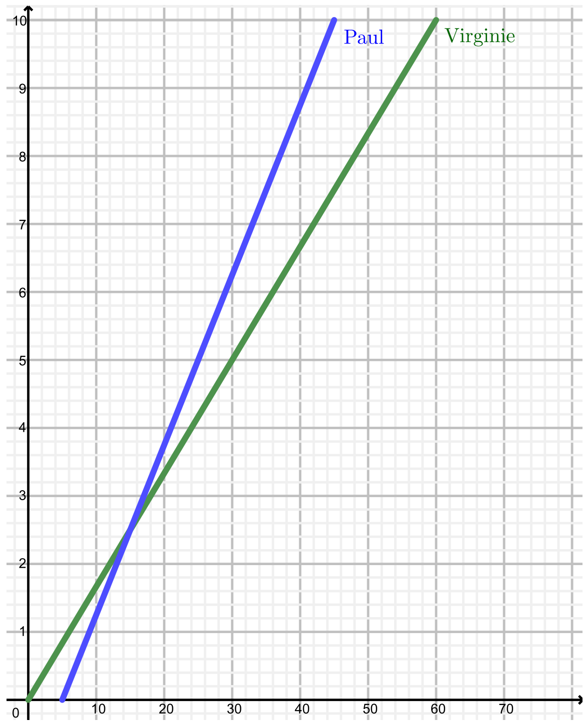
$$e_0 = 0,4a - 717$$

where a represents the year of observation and e_0 the life expectancy at year a.

a) What was a woman's life expectancy in 1900?

b) In what year the life expectancy of a woman reached 65 years?

Exercise 5.19. Paul and Virginie are competing in a 10km race. Virginie starts 5 minutes before Paul, as shown in the graph below.



a) Who arrived first? How many minutes before?

b) How far apart are they when Paul crosses the finish line?

c) What was their respective speed?

Exercise 5.20. A debt of 7'200 francs is amortized at a rate of 300 francs per month. Establish a formula to represent the situation of the debt $C(k)$ as a function of the number of months elapsed ($k \in \mathbb{R}$).

Exercise 5.21. In a shop, printer ink cartridges are sold for 15 francs each. On the web, they are sold for 10 francs each, but you pay 40 francs for the delivery costs, regardless of the number of cartridges purchased.

a) Represent in the same coordinate system the 2 functions determining the price to pay for x cartridges.
We will take 1 unit for 1 cartridge bought on the x-axis and 1 unit for 10 francs on the y-axis.

b) By graphical reading:

   a) Determine the cheapest price for the purchase of six cartridges.

   b) Which option is the most advantageous if you have 80 francs at your disposition?

   c) From how many cartridges is the price on the web lower than the price in the store?

c) Algebraically check the answers obtained above.

Exercise 5.22. The Speedza Company delivers pizzas at home. It offers its sellers a choice of two remuneration models:

— Model 1 : Monthly income of 4'500 francs and 5% commission on sales.

— Model 2 : Monthly income of 4'000 francs and 10% commission on sales.

You're a new seller. What type of remuneration do you choose?

Exercise 5.23. A videoclub offers its customers the following 3 options:

— **Option 1:** 20 francs for an annual subscription and 1 franc per rented DVD.

— **Option 2:** 2 francs per rented DVD and no subscription fee.

— **Option 3:** 70 francs per year whatever the number of DVDs rented.

a) Graphically represent the function determining the price to pay for x DVD, for each of the three options provided, in the same axes system.
We'll take 1 unit for 5 rented DVDs on the x-axis and 1 unit for 10 francs on the y-axis.

b) Read on the graph the most interesting option according to the number of DVDs rented.

Exercise 5.24. A written test has 20 points. The grade of 1 corresponds to 0 point and the grade of 6 to 20 points. Determine the function to calculate the grade y as a function of the number of points x.

Exercise 5.25. A computer technician asks for a flat rate of 30 francs for the transport. What is his hourly rate, knowing that a costumer has paid 390 francs for a reparation requiring 4 and a half hours of intervention?

Exercise 5.26. A car starts driving on a road with a full tank and drives at a constant speed. After 200 km of driving, 40 litres of fuel remain and after 450 km, 15 litres are left. Determine

a) The function that gives the number of litres remaining in the tank as a function of the kilometres covered.

b) The tank capacity.

c) The fuel consumption per 100 km.

d) The maximal distance covered with a full tank.

Exercise 5.27. The volume of a glacier was 125'000 m^3 in 1974 and 16'000 m^3 in 2003. Assuming that we have a linear decrease, estimate in which year this glacier will have totally disappeared.

Exercise 5.28. We made the same rides with two different taxis. With the first one, we paid 8,50 francs for a 2,5 km trip and 15,70 francs for a 5,5 km trip. With the second taxi, for the same distances, we paid 8,25 francs and 16,35 francs. For each taxi, find the function giving the price as a function of the length of the ride. For what distance, is the price the same for both taxis?

Exercise 5.29. A city has installed water treatment factories to supply its citizens with drinking water. The city covers the operating costs by charging a fixed fee and also the water consumed. One of John's neighbours received his bill and for a consumption of 60'000 litres he pays 88 francs. Another neighbour pays 100 francs for a consumption of 75'000 litres. John has not received his bill yet, but he knows that he has consumed 82'000 litres of water.

a) How much will he pay?

b) Determine the value of the fixed fee.

Exercise 5.30. Two car rental agencies have different prices. The first agency charges 300 francs as a fixed fee and then 60 cents for each kilometre covered. We notice that

— for a 250 km ride, the total price is the same in both agencies;

— for a distance of 750 km, the price charged in the second agency is 100 francs more expensive than in the first agency.

Determine the fixed fee and the price per kilometre of the second agency.

Exercise 5.31. We want to subcontract manufacture to three companies that make the following offers:

— A: 150 francs per piece produced.

— B: 75 francs per piece plus a one-off investment of 1'000 francs.

— C: 50 francs per piece plus a one-off investment of 1'500 francs.

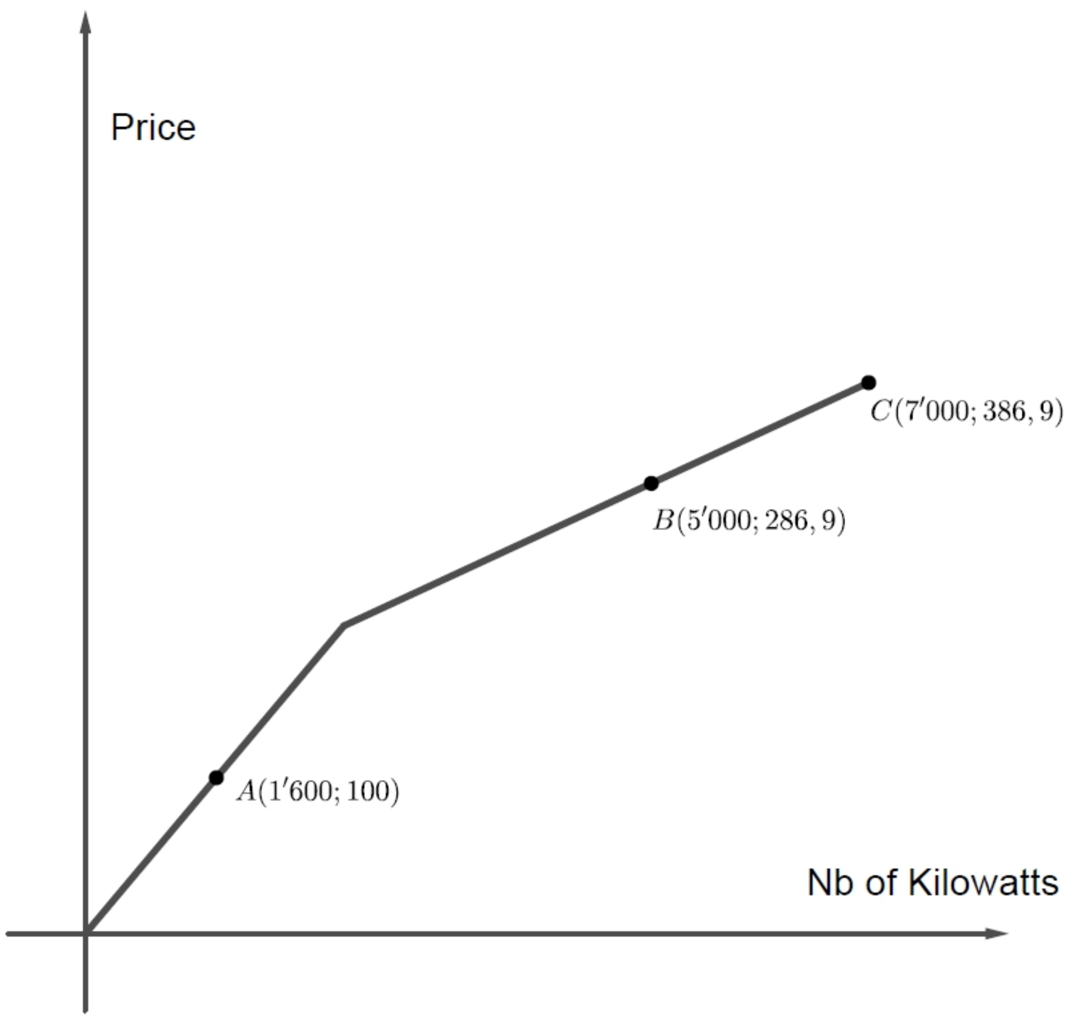
Find the limits for which a company is cheaper than the others.

Exercise 5.32. Here are the prices of a power company

	Price per kilowatt	Monthly fee
From 1 to ...		0 franc
More than ...		36,90 francs

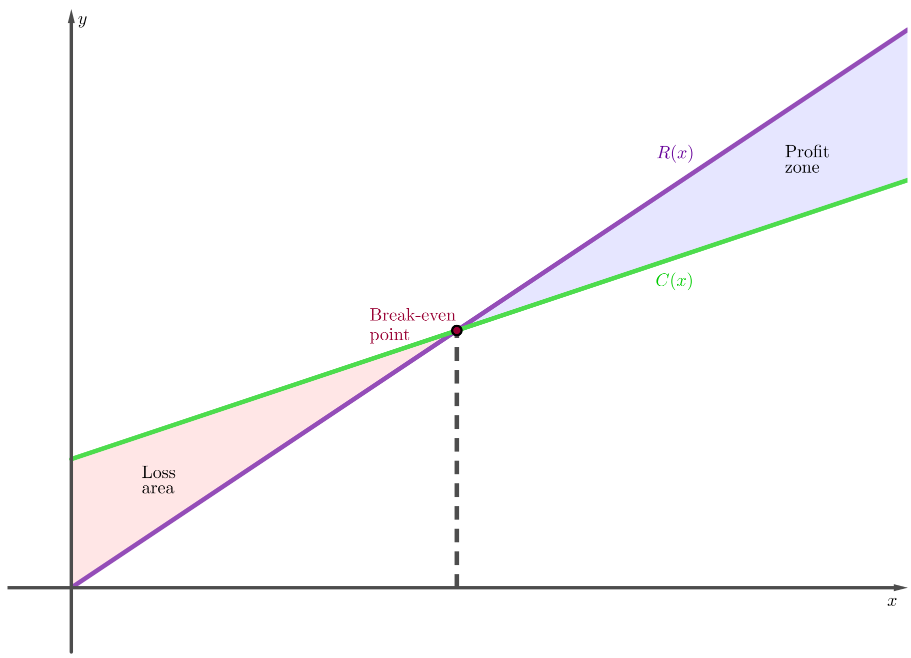
a) Using the graph, determine the price per kilowatt for each situation.

b) From how many kilowatts do you pay a monthly fee?



5.10 Application to economics: Break-even point

One interesting application of the first degree functions is the *break-even point* analysis. The break-even point of a product or firm is determined by the relationship between the cost of producing one (or more) product(s), the volume of production or quantity produced, and the revenue generated by the sale of these products. Production costs are generally made up of fixed and variable costs. Variable costs are often proportional to the quantity produced. Similarly, the income generated by the sale of these objects is usually proportional to the quantity sold. This is graphically illustrated:



If the income is greater than the cost of production, the difference is the profit. If the cost of production is higher than the income, the difference is a loss. If the revenue equals the cost of production, there is neither profit nor loss: it's the break-even point.

To find the break-even point, simply calculate the coordinates of the intersection point of the cost line $C(x)$ and the revenue line $R(x)$.

Example. A flowers vendor sells roses and have a fixed cost of 20 francs a day. He buys his roses 2 francs per piece and sells them for 6 francs.

To determine the number of roses that must be sold to reach the break-even point, the following elements are considered:

— x is the number of sold roses.

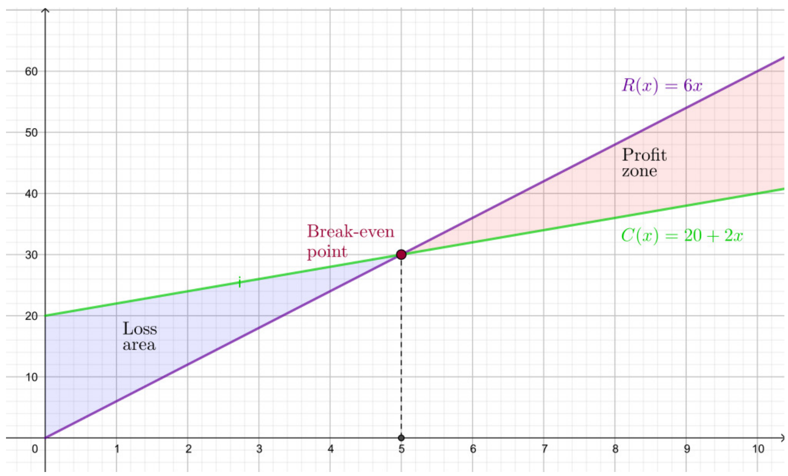
— $C(x) = 2x + 20$ is the production costs.

— $R(x) = 6x$ is the income (or revenue).

We compute the break-even point by putting $C(x) = R(x)$:

$$\begin{aligned} 2x + 20 &= 6x \\ 20 &= 4x \\ x &= 5. \end{aligned}$$

The flowers vendor should sell more than 5 roses to make profit.



What happens if the vendor can now pay 1 franc per rose from another supplier on the outskirts of the city, but has to take a taxi which will cost him an extra 10 francs per day?

The situation is now:

— x is the number of roses sold.

— $C_2(x) = x + 30$ is the production costs.

— $R(x) = 6x$ is the revenue.

We can compute the break-even point by putting $C_2(x) = R(x)$:

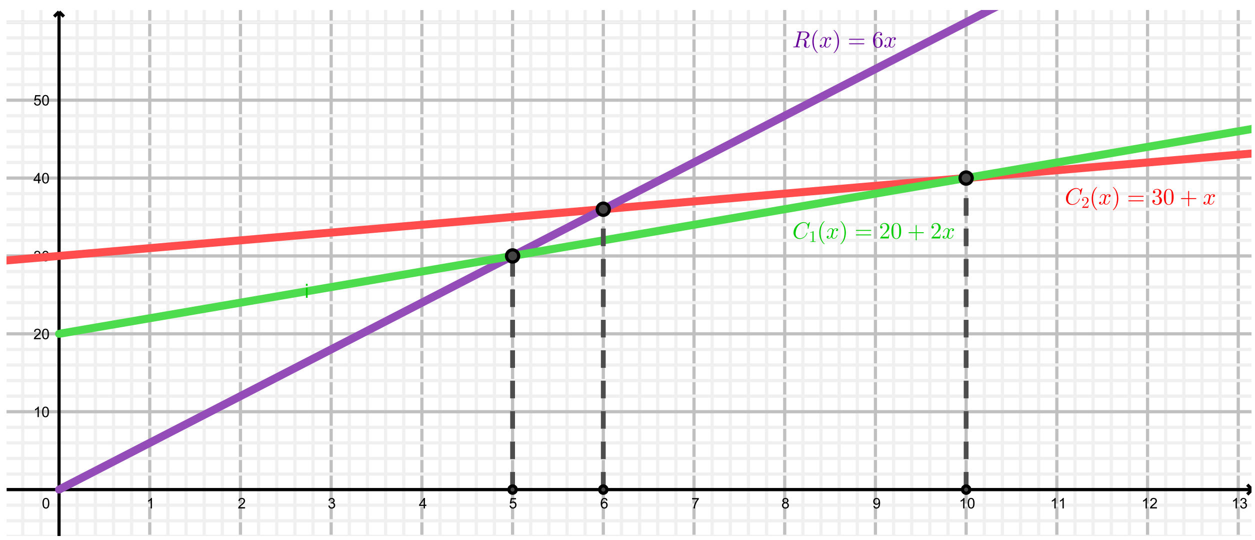
$$\begin{aligned} x + 30 &= 6x \\ 30 &= 5x \\ x &= 6. \end{aligned}$$

The flower vendor has to sell more than 6 roses to make profit.

However, before agreeing to work with this new supplier, he has to determine the number of roses he should sell so that the costs of the new system is lower than the old one. In other words, we look for x such that

$$\begin{aligned} C(x) &= C_2(x) \\ 2x+20 &= x+30 \\ x+20 &= 30 \\ x &= 10. \end{aligned}$$

In conclusion, if the vendor expects to sell more than 10 roses a day, it is in his interest to choose this new supplier. Otherwise, he will have to keep his current supplier.



Exercise 5.33. One balloon maker estimates that it costs 4 francs to make each balloon. In addition, he has calculated that he has a fixed cost of 156 francs per day. If he sells his balloons for 10 francs each, determine his break-even point.

Exercise 5.34. A theatre company was given a contribution of 50'000 francs to develop a theatrical play. Each performance generates 10'000 francs, but the fixed costs (sets, costumes, rehearsals, etc.) are 150'000 francs and the variable costs (salaries of actors, stagehands, etc.) are 8'000 francs per performance.

a) Determine the expression that gives the revenue $R(x)$, the costs $C(x)$ and the profit $P(x)$ as a function of the number of performance x.

b) Does the theatre company break even if it gives 25 performances?

c) Determine the break-even point.

Exercise 5.35. A trader buys articles from a wholesaler at a price of 2 francs each. The trader's fixed operating costs are 148 francs per day. At what price does he have to sell each item to break even at 37 items a day?

Exercise 5.36. A market gardener knows that he can sell all his turnip production by selling them for $0,40$ franc each. He estimates that he has a fixed cost of 100 francs per day and that it costs him $0,20$ franc to produce each turnip.

a) Compute the break-even point.

b) A seller of heavy equipment proposes to the gardener the purchase of a machine that will reduce the cost of production to $0,10$ franc per turnip but will increase his fixed costs to 180 francs per day. Give the new break-even point, then determine the number of turnips the producer will have to sell for the new system to be more advantageous.

Exercise 5.37. A publisher decides to publish a mathematics book. The costs he has to cover are made up of fixed costs (composition, editing, ...) which are 11'000 francs and variable costs (printing, copyrights, ...) which amount to 9 francs per volume.

a) If he sells his books for 20 francs a copy, what's his break-even point?

b) A new method of document composition reduces the fixed costs to 9'500 francs. However, in this case, the printing process increases the variable costs to 10 francs per book. Calculate the new break-even point and the number of books to be sold to help him choose a process.

Exercise 5.38. A group of friends are recording a DVD. The fixed costs represent a fixed sum to be paid regardless of the number of DVDs produced (electricity, heating, equipment rental, ...). It is therefore theoretically due even in the event of a last-minute problem and nothing is produced. The fixed costs here amount to 28'000 francs. The variable costs per DVD represent the increase in total costs each time an additional DVD is produced. The total production cost for a given quantity is the sum of the fixed and variable costs. Producing 500 units costs 30'150 francs. DVDs are sold for 25 francs each.

a) Calculate the amount of variable costs per DVD.

b) How much would it cost to produce 2'000 DVDs?

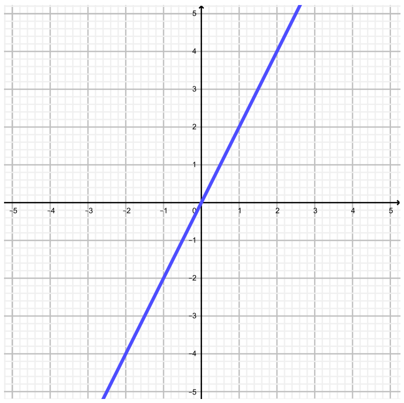
c) Calculate the break-even point (the number of DVDs and also the number of francs).

d) Determine how many DVDs need to be sold in order to make a profit of 50'000 francs.

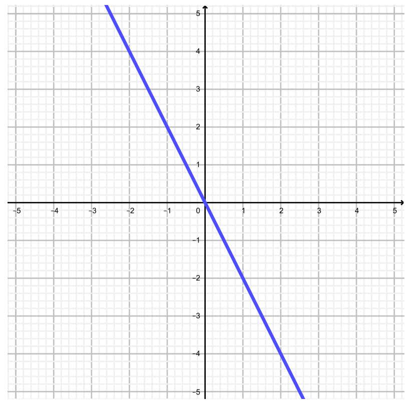
5.11 Solutions

Exercise 5.1.

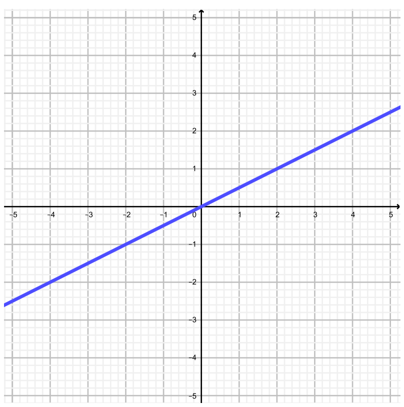
a)



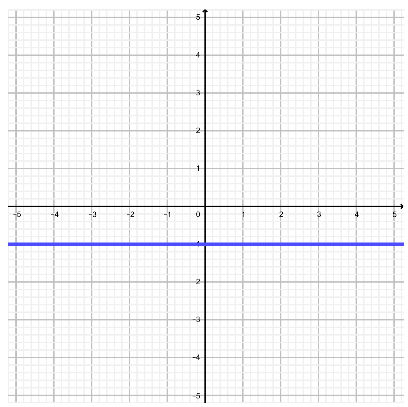
b)



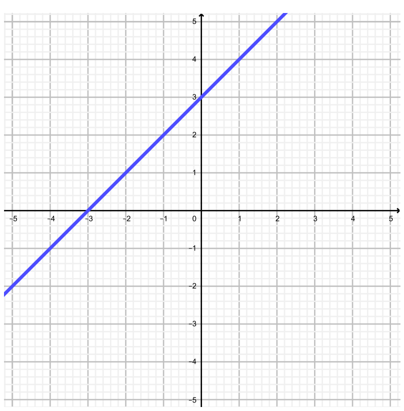
c)



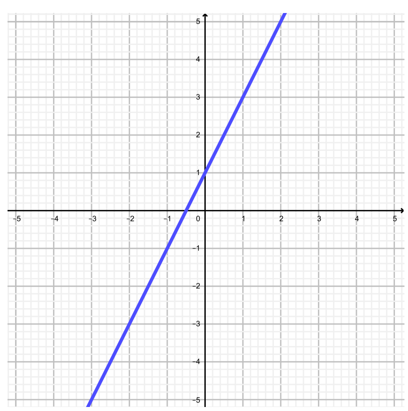
d)



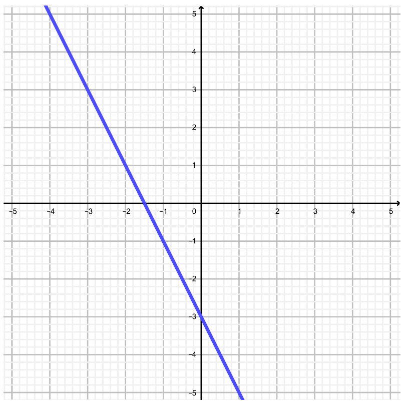
e)



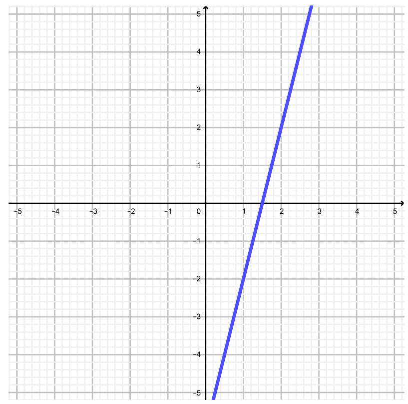
f)



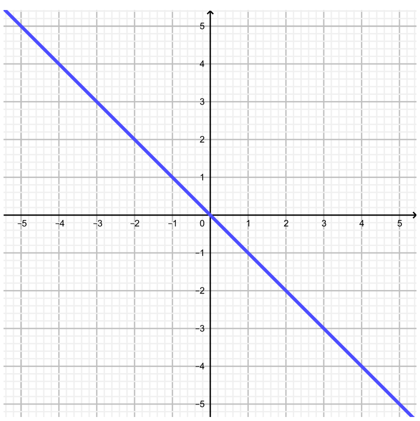
g)



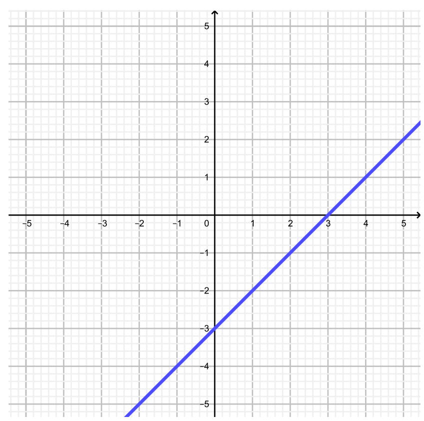
h)



i)

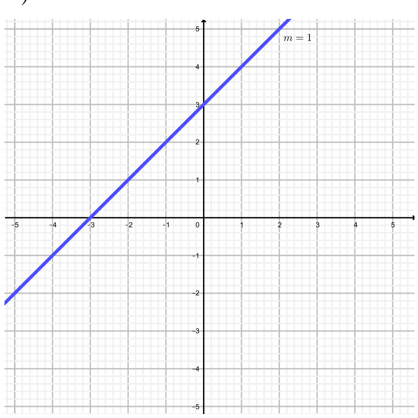


j)



Exercise 5.2.

a)



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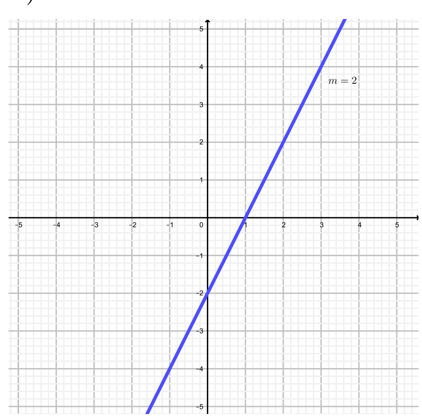

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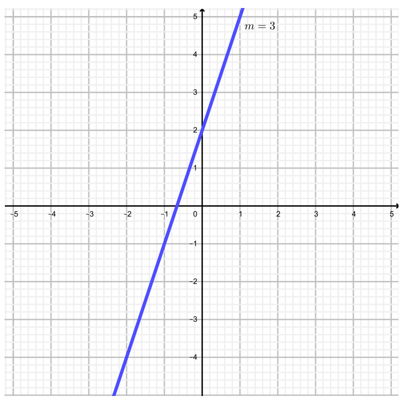

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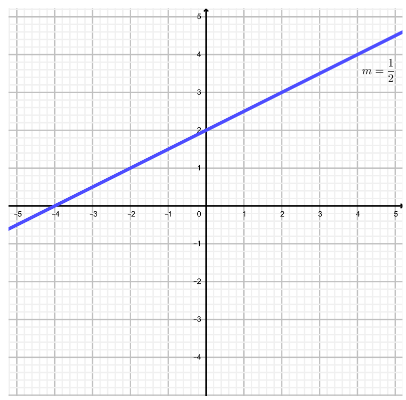

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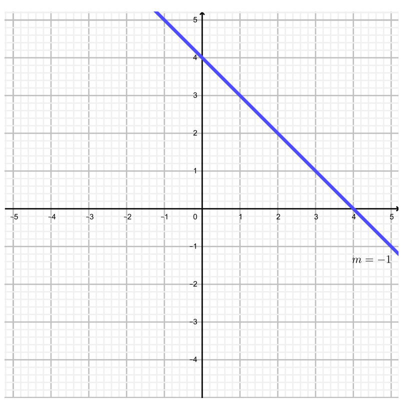

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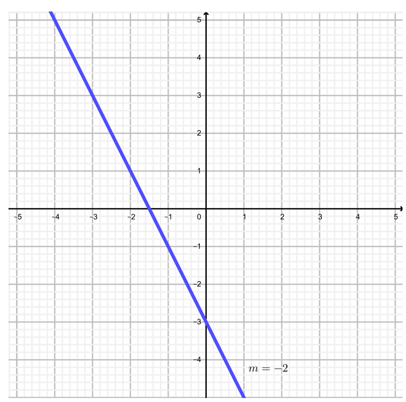

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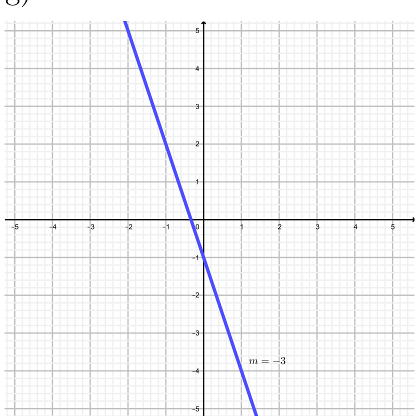

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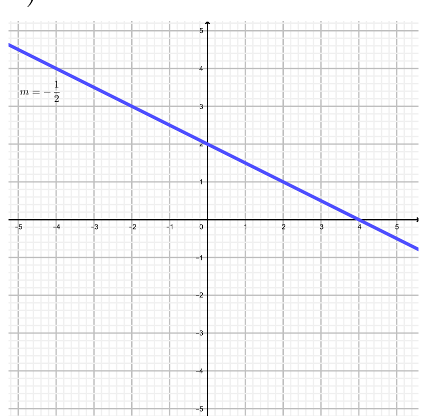
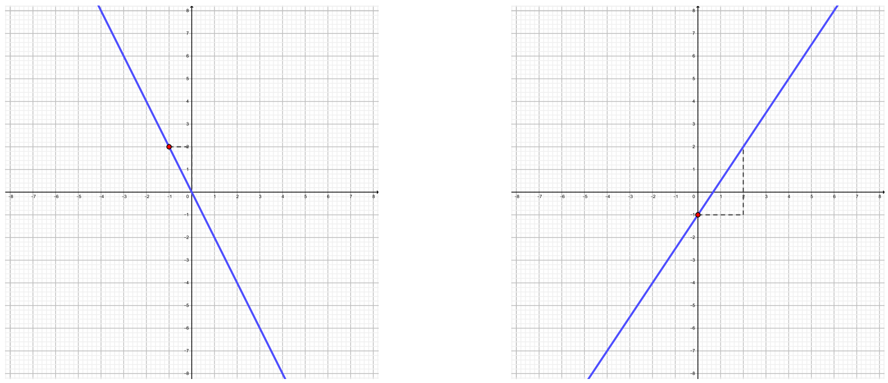
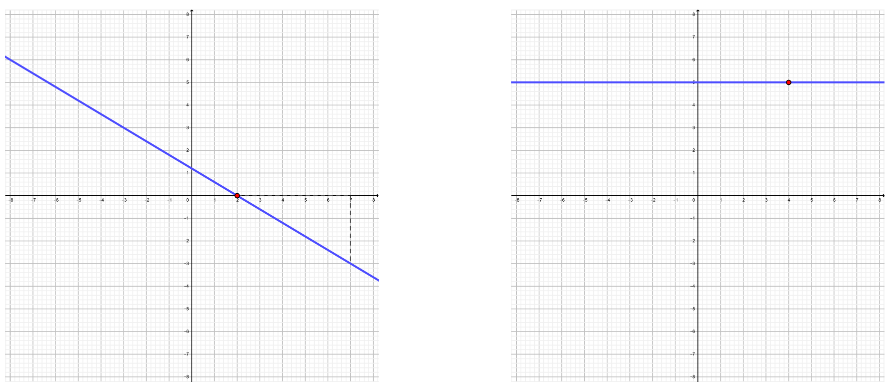


Exercise 5.3.

a) b)



c) d)



Exercise 5.4. $y = -3x + 19$.

Exercise 5.5.

a) $y = 3x + 2$ b) $y = 5x - 4$

c) $y = \frac{3}{4}x - 6$ d) $y = -\frac{4}{7}x - \frac{9}{7}$

Exercise 5.6.

$f(x) = 2x - 3$.

$g(x) = -x + 4$.

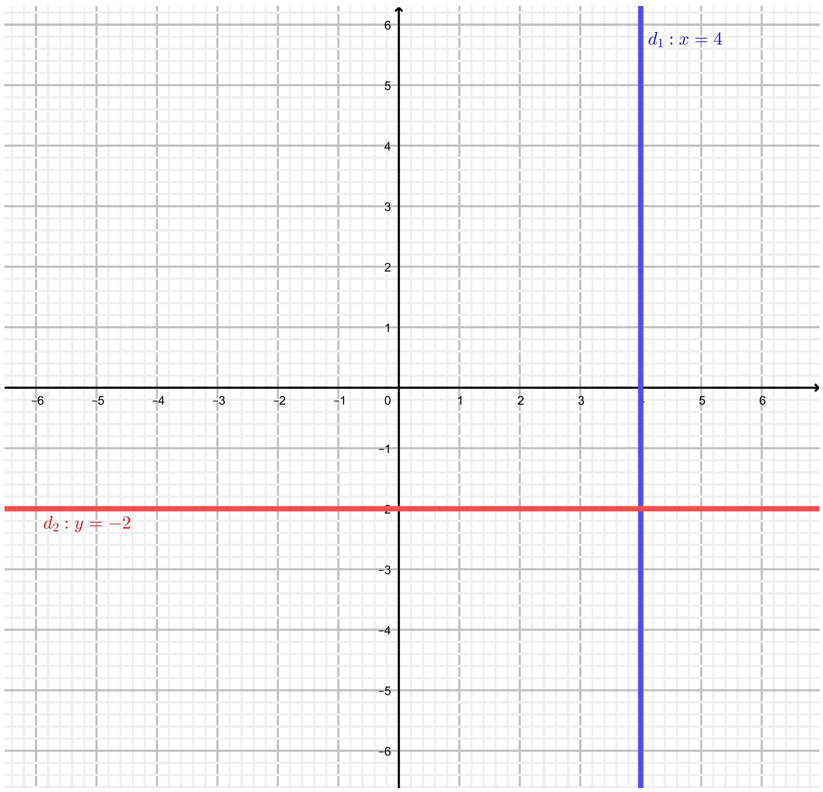
$h(x) = \frac{3}{5}x + 2$.

$i(x) = -\frac{4}{7}x$.

$j(x) = 3$.

Exercise 5.7. $x = -4$.

Exercise 5.8.



Exercise 5.9.

a) $y = -3x$.

b) $y = \frac{3}{2}x - 8$.

c) $y = \frac{1}{3}x$.

d) $y = -\frac{1}{2}x + 4$.

Exercise 5.10. $y = -x - 2$.

Exercise 5.11. $y = \frac{2}{3}x - 7$.

Exercise 5.12.

a) $I(1;4)$
b) $I(2;0)$
c) $I(3;2)$
d) $I(3;13)$
e) $I(-2;12)$
f) $I(0;0)$
g) No intersection point
h) Infinity of intersection points

Exercise 5.13. $I(-4;-3)$.

Exercise 5.14. They form a triangle.

Exercise 5.15.

a) $(2;0)$ and $(0;-4)$.

b) $(2;0)$ and $\left(0;\dfrac{6}{5}\right)$.

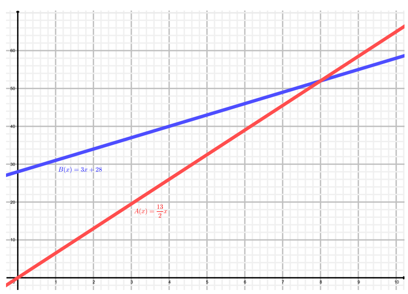
Exercise 5.16.

a)

Number of matches	4	6	10	15
Price paid with option A	26	39	65	$97,50$
Price paid with option B	40	46	58	73

b) $A(x) = \dfrac{13}{2}x$, $B(x) = 3x + 28$.

c)



d) a) For 8 matches, the price is therefore 52 francs.
   b)

Exercise 5.17.

a) $2,\overline{2}$°C.

b) -40°C $= -40$ F.

c) 320 F.

Exercise 5.18.

a) 43 years old.

b) In 1955.

Exercise 5.19.

a) Paul: 15 minutes earlier than Virginie.

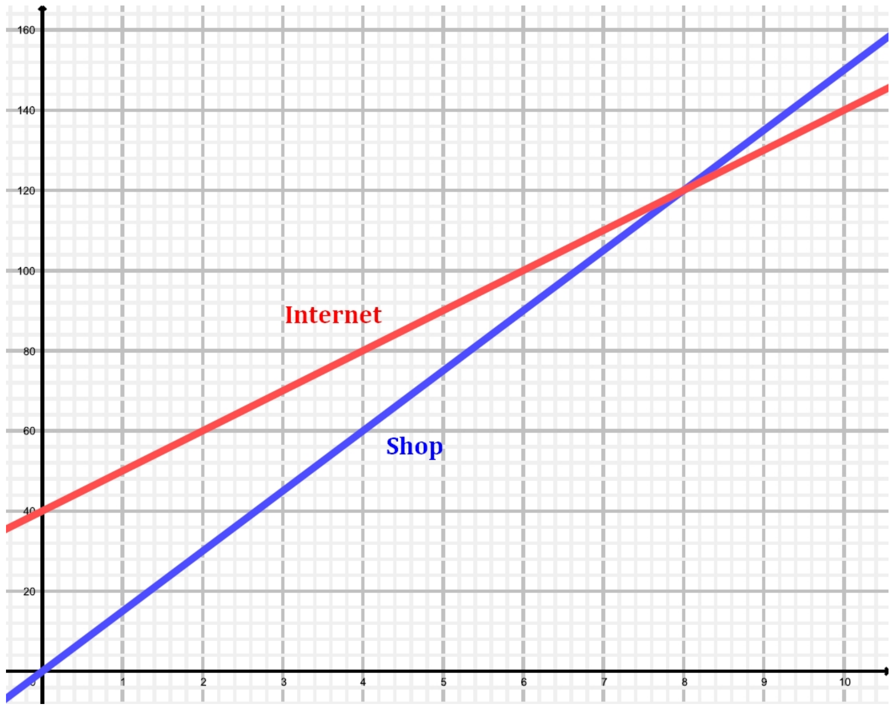
b) $2,5$ km.

c) Paul: $15\frac{\text{km}}{\text{h}}$, Virginie: $10\frac{\text{km}}{\text{h}}$.

Exercise 5.20. $C(k) = 7'200 - 300k$.

Exercise 5.21.

a)



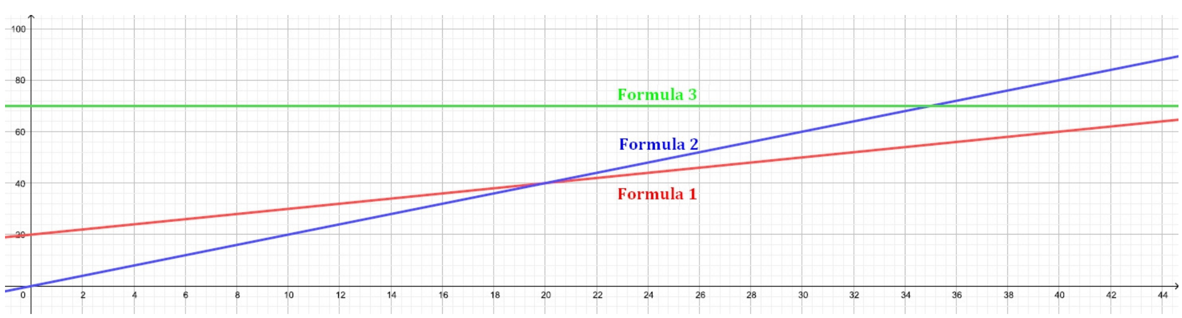
b)
  a) 90 francs (shop).
  b) Shop.
  c) From 8 ink cartridges.

c)

Exercise 5.22. From 1 to 10'000 francs: model 1. Otherwise model 2.

Exercise 5.23.

a)



b) Between 0 and 20 DVDs rented: Formula 2.
Between 20 and 50 DVDs rented: Formula 1.
From 50 rented DVDs: Formula 3.

Exercise 5.24. $y = \frac{1}{4}x + 1$.

Exercise 5.25. 80 francs per hour.

Exercise 5.26.

a) $f(x) = -\frac{1}{10}x + 60$.

b) 60 litres.

c) 10 litres per 100 km.

d) 600 km.

Exercise 5.27. During year 2007.

Exercise 5.28. $y_1 = 2,4x + 2,5$, $y_2 = 2,7x + 1,5$, $y_1 = y_2$ for a ride of $x = 3,33$ km.

Exercise 5.29.

a) $105,60$ francs.

b) 40 francs.

Exercise 5.30. Fixed costs: 250 francs and cost per kilometer: 80 cents.

Exercise 5.31. Up to 13 pieces: A. From 14 to 20 pieces: B. More than 20 pieces: C.

Exercise 5.32.

a) $6,25$ cents and 5 cents.

b) $y = 0,0625x$, $y = 0,05x + 36,9$. From $x = 2'952$ kilowatts.

Exercise 5.33. 26 balloons.

Exercise 5.34.

a) $R(x) = 50'000 + 10'000x$, $C(x) = 150'000 + 8'000x$ and $B(x) = 2'000x - 100'000$.

b) No.

c) 50 performances.

Exercise 5.35. 6 francs.

Exercise 5.36.

a) 500 turnips.

b) The new break-even point is 600 turnips. He should accept this offer if he sells more than 800 turnips.

Exercise 5.37.

a) 1'000 books.

b) 950 books is the break-even point. He should accept this new process if he sells up to 1500 books.

Exercise 5.38.

a) The variable costs are 4,3 francs per DVD.

b) 36'600 francs.

c) Break-even point: $x \cong 1'353$ DVDs and $y \cong 33'816$ francs.

d) 3'768 DVDs.

5.12 Chapter objectives

At the end of this chapter, the student should be able to

5.1 ☐ Sketch the graph of a first degree function using a table a values.
5.2 ☐ Compute the slope of a line.
5.3 ☐ Sketch the graph of a first degree function using the slope and the y-intercept.
5.4 ☐ Determine the functional notation of a function from its graph, from two points or from a slope and a point.
5.5 ☐ Sketch a vertical or horizontal line.
5.6 ☐ Determine the equation of a line parallel to another one.
5.7 ☐ Determine the equation of a line perpendicular to another one.
5.8 ☐ Compute the coordinates of the intersection point of two lines.
5.9 ☐ Compute the coordinates of the intersection points of a line with the axes.
5.10 ☐ Solve a problem involving first degree functions.
5.11 ☐ Solve a problem related to economics.

Chapter 6

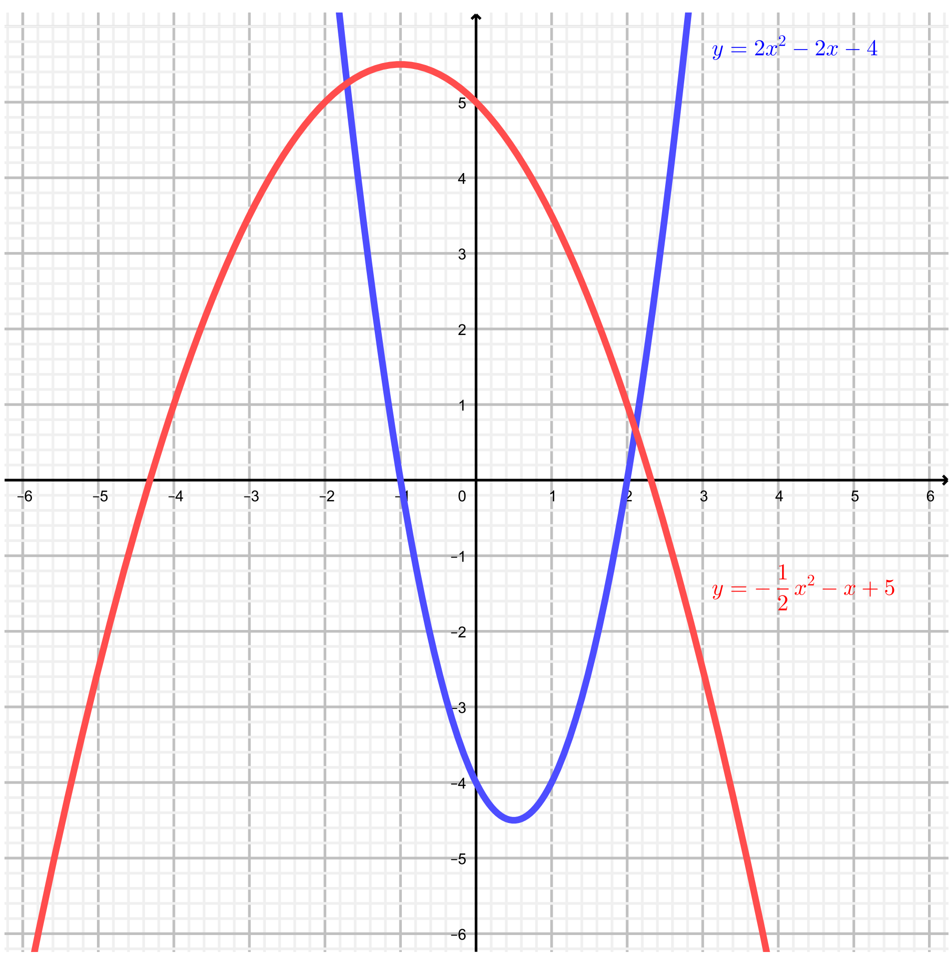
Second degree functions

6.1 Definition

Definition. A *second degree function* (or *quadratic function*) is any function that can be written in the form

$$\boxed{f(x) = ax^2 + bx + c.}$$

Graphically, such a function is a *parabola*.



Coefficient a represents the *convexity* of the parabola.

— If $a > 0$, then the function is *convex*. "It's smiling !".

— If $a < 0$, then the function is *concave*. "It's not happy".

— If $a = 0$, then it's no longer a parabola but a line.

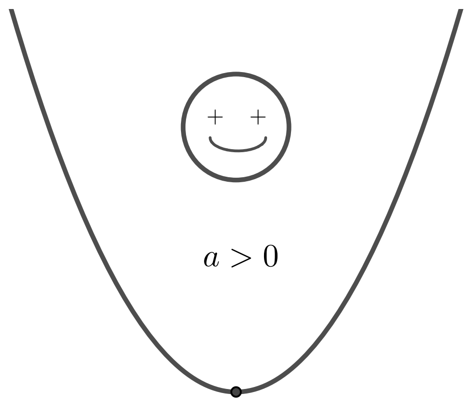


Figure 6.1: f convex.

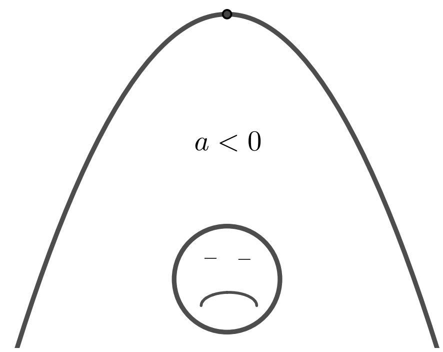


Figure 6.2: f concave.

The letter c represents the *y-intercept* of the parabola, just like the letter h was for lines.

Exercise 6.1. Sketch the graph of the following functions.

a) $f(x) = x^2 - 4x$ b) $f(x) = -x^2 + 4$

c) $f(x) = 2x^2 - 4x - 2$ d) $f(x) = -\frac{1}{2}x^2 - x - 4$

Exercise 6.2. Without any graph, answer the following questions.

a) Is the point $P(1; 6)$ on the graph of $f(x) = x^2 + 8x - 3$?

b) Is the point $P(2; 7)$ on the graph of $f(x) = x^2 + 8x - 11$?

c) The point P whose abscissa $x = 3$ is on the graph of $f(x) = x^2 - 7x + 3$. Determine its ordinate y.

6.2 Properties of parabolas

Let $\mathcal{P}$ be a *parabola* of equation

$$\mathcal{P} : y = ax^2 + bx + c$$

with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Curves

The larger $|a|$ is, the tighter the curve is.

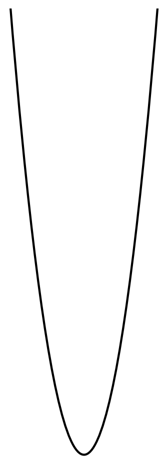


Figure 6.3: $|a|$ large.

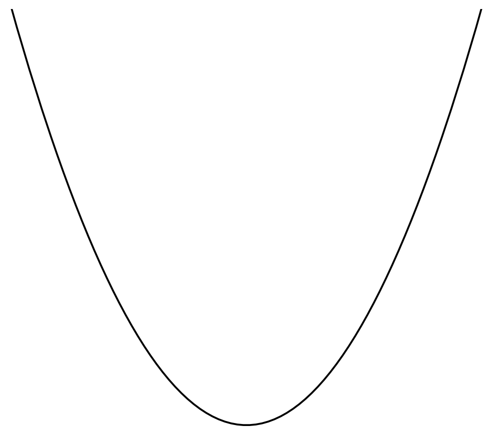
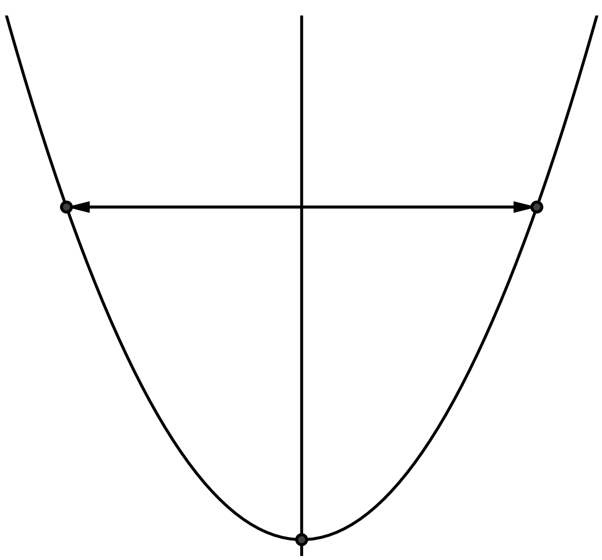


Figure 6.4: $|a|$ small.

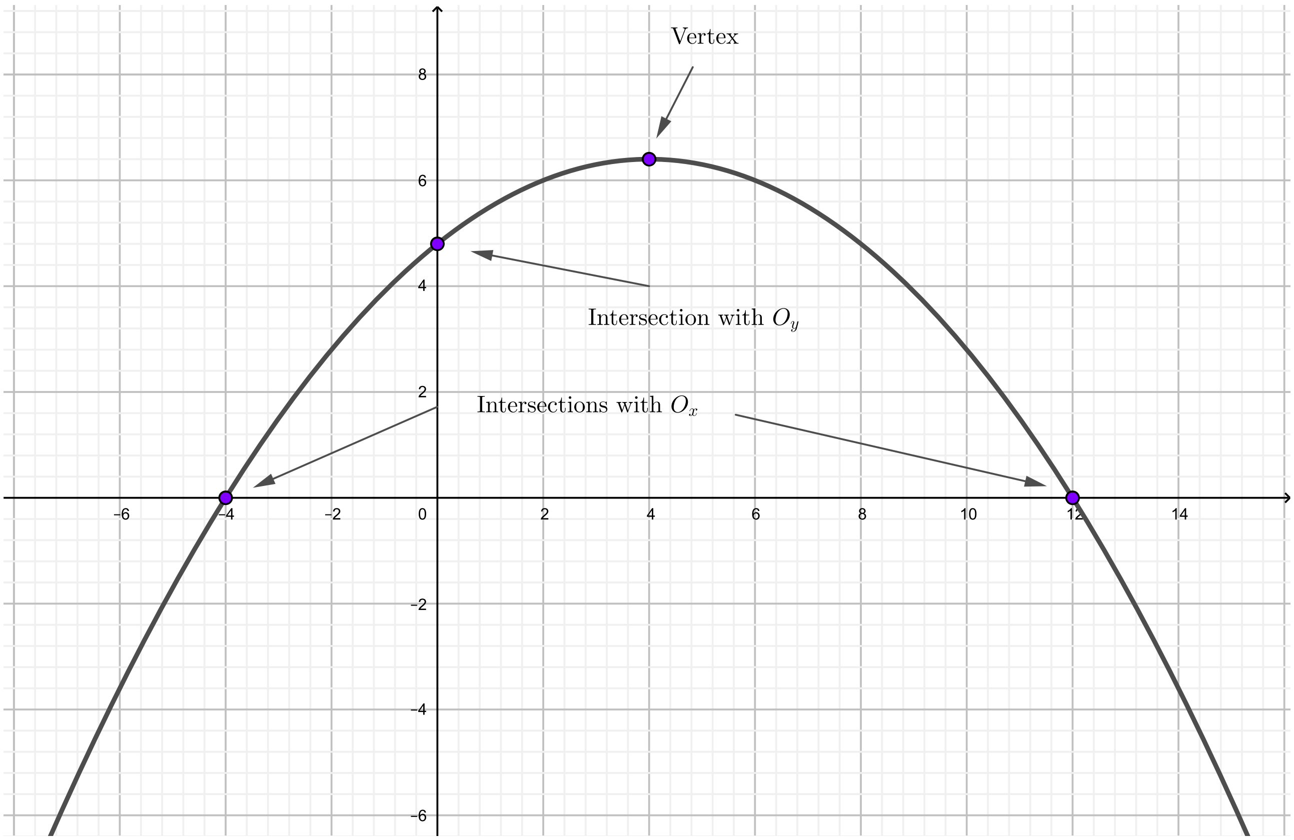
Symmetry

The curve is symmetric with respect to a vertical line passing through its vertex.



Intersection with the axes and vertex

Parabolas have important points such as the *vertex* and the *intersection points with the axes*.



How do we compute those points?

1. **Intersections with the x-axis.**
   We must find the possible zeros of the function (x_1 and x_2). As in the case of the lines, we therefore put $y = 0$ and we find the possible value(s) of x.

   Example. Let $f(x) = 3x^2 + 3x - 18$ be a quadratic function.
   We solve

$$\begin{array}{rcl|l} 3x^2 + 3x - 18 & = & 0 & :3 \\ x^2 + x - 6 & = & 0 & \text{Factorize} \\ (x+3)(x-2) & = & 0 & \end{array}$$

   We get $x_1 = -3$ and $x_2 = 2$. We would have found the same result using the quadratic formula.

   We can deduce the following intersection points $(-3; 0)$ and $(2; 0)$.

2. **Intersection with the y-axis.**
   We put $x = 0$ and we look for the value of y.

   Example. Let's see with $f(x) = 3x^2 + 3x - 18$.
   We put $x = 0$, and we get

$$y = 3 \cdot 0^2 + 3 \cdot 0 - 18 = -18.$$

   So, the intersection point is $(0; -18)$.

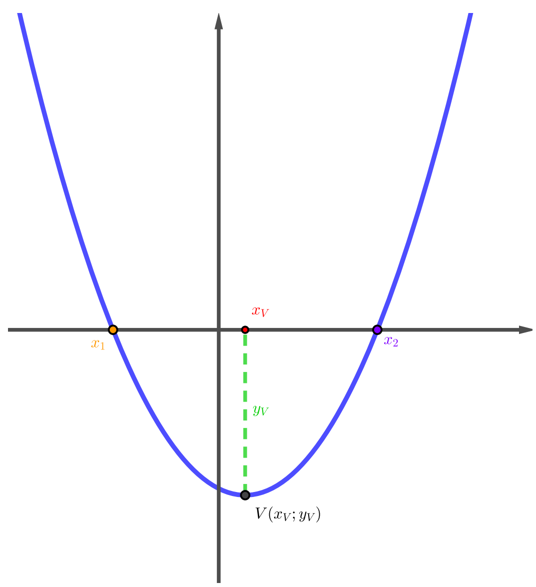
3. **Vertex $V(x_V; y_V)$.**

   Theorem. *The vertex of a second degree function is a minimum if $a > 0$, and a maximum if $a < 0$. Its coordinates are given by:*

$$\boxed{x_V = \frac{x_1 + x_2}{2} = \frac{-b}{2a} \text{ and } y_V = f(x_V).}$$

   Remark. We therefore find y_V by replacing x in the functional notation by the value x_V just found.

   Proof. The vertex of the parabola is located right in the middle of the possible zeros x_1 and x_2.



   Its abscissa x_V is given by

$$\begin{aligned} x_V &= \frac{x_1+x_2}{2} &= \frac{\frac{-b-\sqrt{\Delta}}{2a}+\frac{-b+\sqrt{\Delta}}{2a}}{2} \\ &= -\frac{2b}{2a} \cdot \frac{1}{2} &= -\frac{b}{2a} \end{aligned}$$

   When x_V is known, we can deduce the y-coordinate y_V with

$$y_V = f\left(-\frac{b}{2a}\right).$$

□

Example. With $f(x) = 3x^2 + 3x - 18$.
We have

$$x_V = \frac{-b}{2a} = \frac{-3}{2 \cdot 3} = -\frac{1}{2} \text{ and } y_V = f\left(-\frac{1}{2}\right) = 3 \cdot \left(-\frac{1}{2}\right)^2 + 3 \cdot \left(-\frac{1}{2}\right) - 18 = -\frac{75}{4}.$$

We can compute x_V differently:

$$x_V = \frac{x_1 + x_2}{2} = \frac{-3+2}{2} = -\frac{1}{2}.$$

The vertex is then

$$V\left(-\frac{1}{2}; -\frac{75}{4}\right).$$

Exercise 6.3. Determine the coordinates of the intersection points with the axes and the vertex of the following functions. Determine if the vertex is a maximum or a minimum.

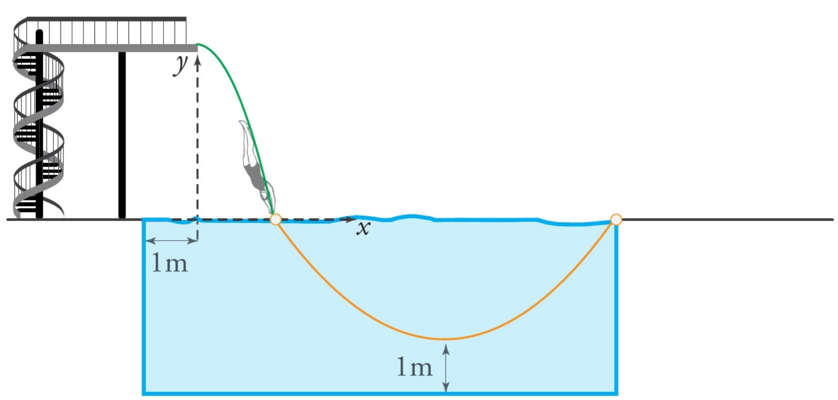
a) $f(x) = x^2 - 2x - 3$
b) $f(x) = -x^2 + 5x$
c) $f(x) = 3x^2 + 3$
d) $f(x) = -2x^2 + 5x - 4$

Exercise 6.4. A diver jumps from a diving board and describes a parabolic trajectory that can be modeled by the function f defined by

$$f(x) = -1,25(x^2 - 4).$$

Once in the water, he follows another parbolic trajectory described by the function

$$g(x) = 0,6x^2 - 4,8x + 7,2.$$



Determine

a) How high is the diving board.
b) The distance between him and the edge of the pool when he enters the water.
c) How long is the pool.
d) How deep is the pool.

6.3 Different forms of functional notation

1. **Polynomial form :** $y = ax^2 + bx + c$.

   Advantage: we know the intersection point with the y-axis.

   Example.

   (a) With $y = x^2 - 4x + 1$, we can deduce the intersection point with the y-axis: $(0; 1)$.

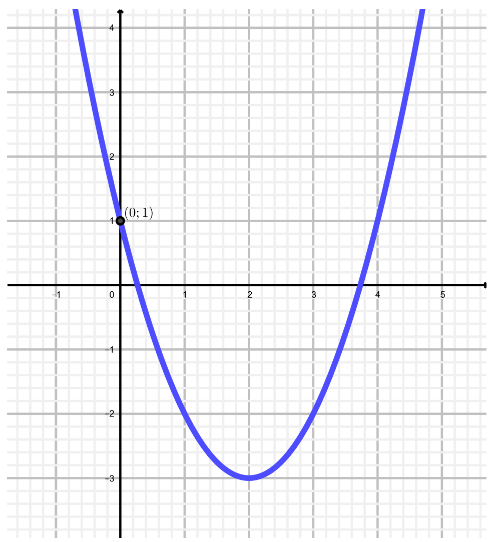


Figure 6.5: Graph of $y = x^2 - 4x + 1$.

   (b) With $y = -2x^2 + 6x - 4$, we can deduce the intersection point with the y-axis: $(0; -4)$.

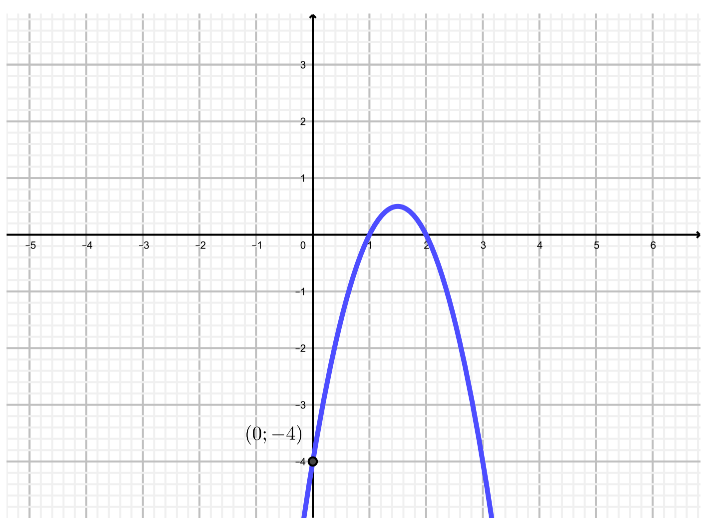


Figure 6.6: Graph of $y = -2x^2 + 6x - 4$.

2. **Standard form :** $y = a(x - x_V)^2 + y_V$.

   Here, the letter a is the same letter as in the polynomial form. Hence, it gives the convexity.

   Advantage: we know the vertex. x_V and y_V are its coordinates.

   Example.

   (a) With $y = 3(x-2)^2 + 5$, we can deduce the coordinates of the vertex: $V(2; 5)$.

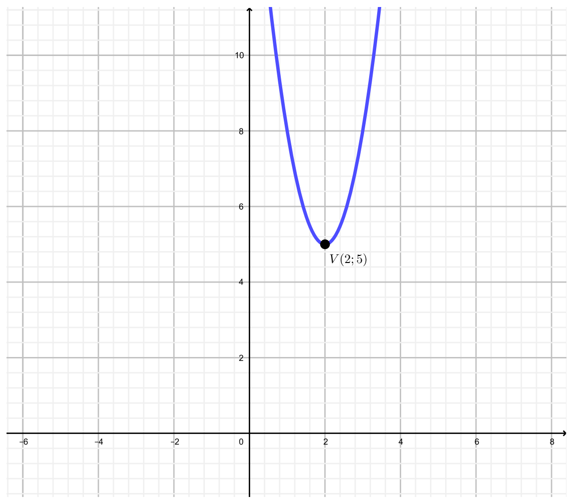


Figure 6.7: Graph of $y = 3(x-2)^2 + 5$.

   (b) With $y = -2(x+3)^2 - 3$, that we can also write $y = -2(x - (-3))^2 - 3$. We can deduce the coordinate of the vertex: $V(-3; -3)$.

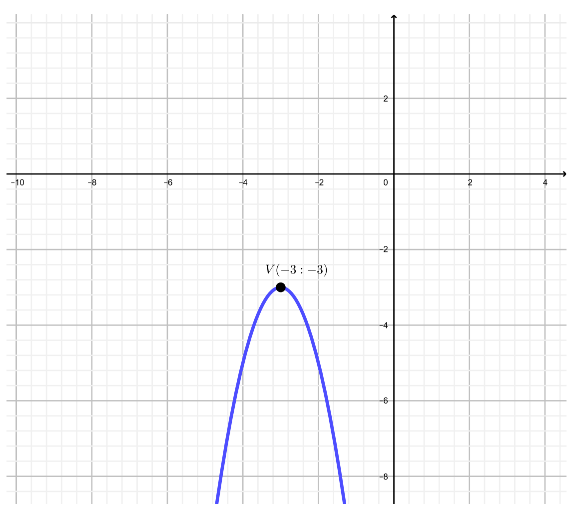


Figure 6.8: Graph of $y = -2(x+3)^2 - 3$.

3. **Factored form :** $y = a(x - x_1)(x - x_2)$.

   The letter a is the same and still gives the convexity.

   Advantage: we know the intersection points with the x-axis. x_1 and x_2 are the zeros of the function (in other words, the x-coordinates of both intersection points with the x-axis).

   Example.

   (a) Let $y = (x - 2)(x - 4)$ be a function. We can deduce the intersection points with the x-axis: $(2; 0)$ and $(4; 0)$.

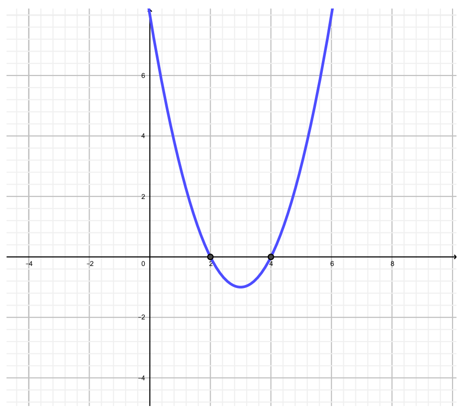


Figure 6.9: Graph of $y = (x - 2)(x - 4)$.

   (b) With $y = 3(x + 5)\left(x - \dfrac{7}{2}\right)$, that also can be written as $y = 3(x - (-5))\left(x - \dfrac{7}{2}\right)$,

   We can deduce the intersection points with the x-axis: $(-5; 0)$ and $\left(\dfrac{7}{2}; 0\right)$.

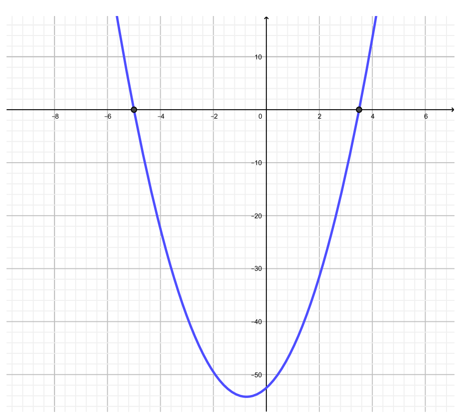
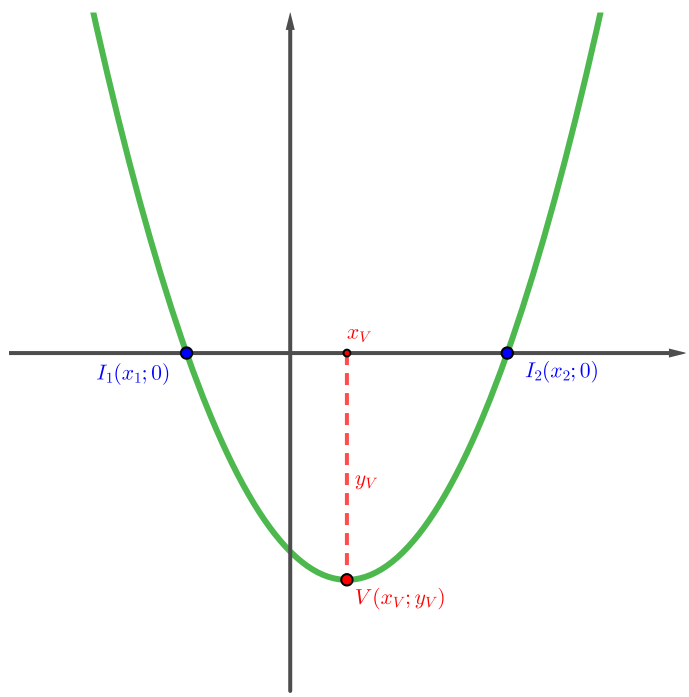


Figure 6.10: Graph of $y = 3(x + 5)\left(x - \dfrac{7}{2}\right)$.

Each form has its own advantages and disadvantages.

The table below summarizes the properties of the three forms.

Form	Functional notation	Vertex	Intersection points with O_x
Polynomial	$f(x) = ax^2 + bx + c$	$V(x_V; y_V)$ where $x_V = -\dfrac{b}{2a}$	When solving $f(x) = 0$
Standard	$f(x) = a(x - x_V)^2 + y_V$	$V(x_V; y_V)$	When solving $f(x) = 0$
Factored	$f(x) = a(x - x_1)(x - x_2)$	$V(x_V; y_V)$ where $x_V = \dfrac{x_1 + x_2}{2}$	$I_1(x_1; 0)$ and $I_2(x_2; 0)$



Exercise 6.5. Determine the coordinates of the vertex of each of the following parabolas and specify whether this is a maximum or minimum.

a) $y = 5(x - 4)^2 - 3$ b) $y = (x - 1)^2 + 2$
c) $y = 4(x - 2)^2$ d) $y = -2(x + 3)^2$
e) $y = (x - 1)(x + 1)$ f) $y = 5(x - 4)(x + 3)$
g) $y = -2(x + 3)(x - 4)$ h) $y = -0,5(x - 10)^2$
i) $y = -0,5x(x - 10)$ j) $y = 3x(x + 5)$
k) $y = 4x^2 - 5$ l) $y = 3x^2 + 7$

Exercise 6.6. Determine the coordinates of the intersection points with the axes for the following parabolas

a) $y = 2(x-3)(x-4)$ b) $y = 2(x-1)^2 - 8$

c) $y = -\frac{1}{3}(x-5)(x+3)$ d) $y = -3(x+2)^2 - 6$

6.4 Graph of a second degree function

For a line, two points are enough to define it. This is not always the case for a second degree function. It is necessary to calculate the coordinates of the intersection points with the axes and the vertex. If this is not sufficient, a table of values can be built.

Example. Let's sketch the parabola $\mathcal{P} : y = -2(x-2)(x+4)$.

The first thing to do is to identify whether the parabola is given in a known form. Here we notice that this is the factored form. The advantage of the factored form is that the zeros of the function can be deduced from it:

$$x_1 = 2 \text{ and } x_2 = -4.$$

The intersection points with the x-axis are therefore: $(2; 0)$ and $(-4; 0)$.

To find the intersection point with the y-axis, we just replace x by 0.

$$y = -2(0-2)(0+4) = -2(-2)(4) = 16.$$

Hence, the intersection point with the y-axis is given by the point $(0; 16)$.

For the vertex, we can choose the appropriate method to first find its coordinate x_V. Since we already have x_1 and x_2, we can calculate

$$x_V = \frac{x_1 + x_2}{2} = \frac{2 + (-4)}{2} = -1.$$

To find y_V, we just replace x by the value of x_V in the functional notation.

$$y_V = -2(-1-2)(-1+4) = -2(-3)(3) = 18.$$

The vertex is therefore $V(-1; 18)$.

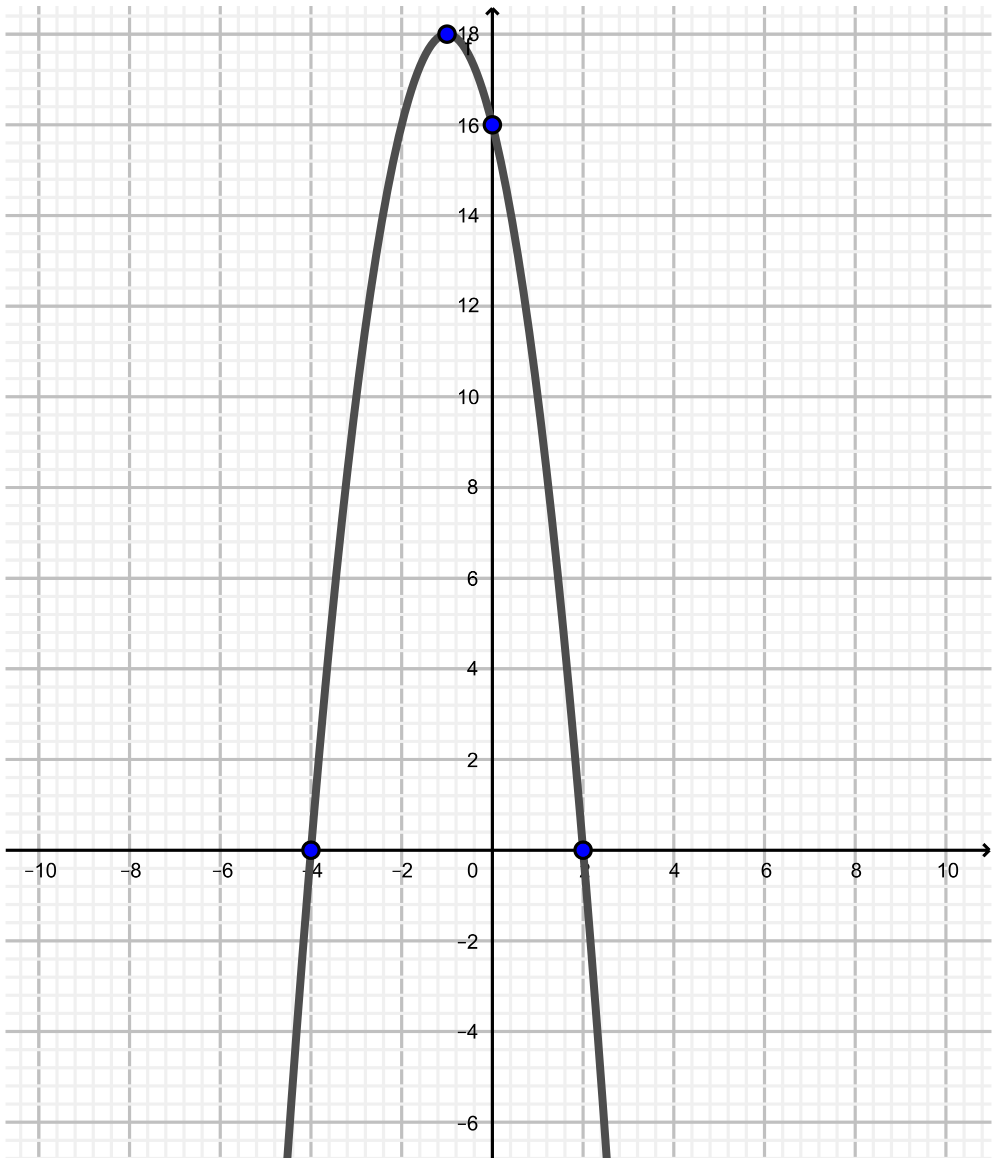


Figure 6.11: Graph of $y = -2(x-2)(x+4)$.

How to find the equation of a parabola passing through some given points?

Depending on the nature of the known points, the appropriate form is chosen. A last point is then used to find the value of the coefficient a.

Example. Let's find a parabola passing through the points $(-3; 0)$, $(1; 0)$ and $(3; 4)$.

We notice here that we have the points of intersection with the x-axis. In other words, we know that $x_1 = -3$ and $x_2 = 1$ (or vice versa). Knowing that, we will choose the factored form

$$y = a(x - x_1)(x - x_2),$$

which can be now written

$$\begin{aligned} y &= a(x - (-3))(x - 1) \\ y &= a(x + 3)(x - 1). \end{aligned}$$

In order to find the value of a, we use the point $(3;4)$. Indeed, we know that when $x = 3$, $y = 4$, so we can temporarily replace it in the equation to get only one unknown (which is a).

$$\begin{array}{rcl|l} 4 &=& a(3+3)(3-1) & \\ 4 &=& 12a & :12 \\ a &=& \dfrac{1}{3}. & \end{array}$$

Hence, the equation of the parabola is

$$y = \frac{1}{3}(x+3)(x-1).$$

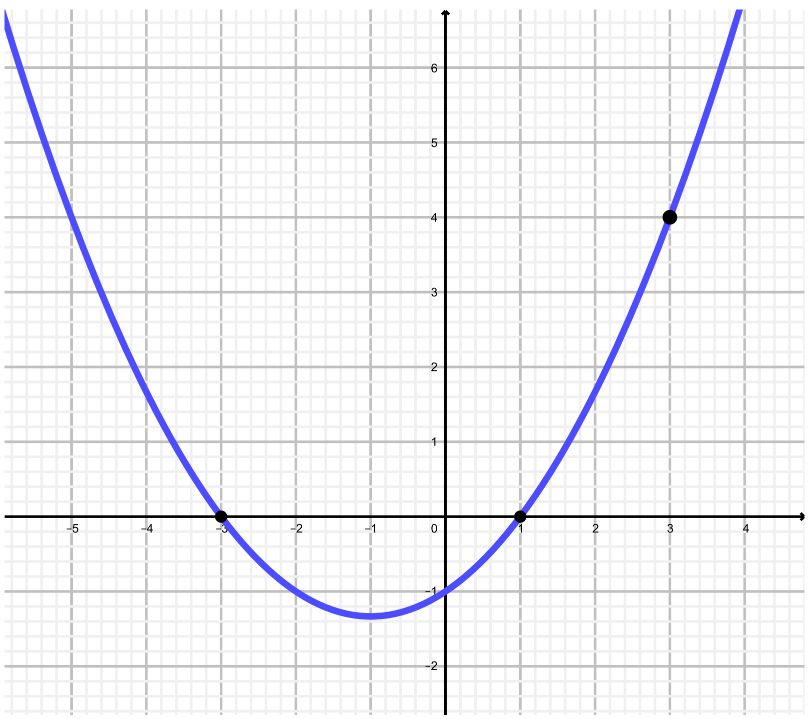


Figure 6.12: Graph of $y = \dfrac{1}{3}(x+3)(x-1)$.

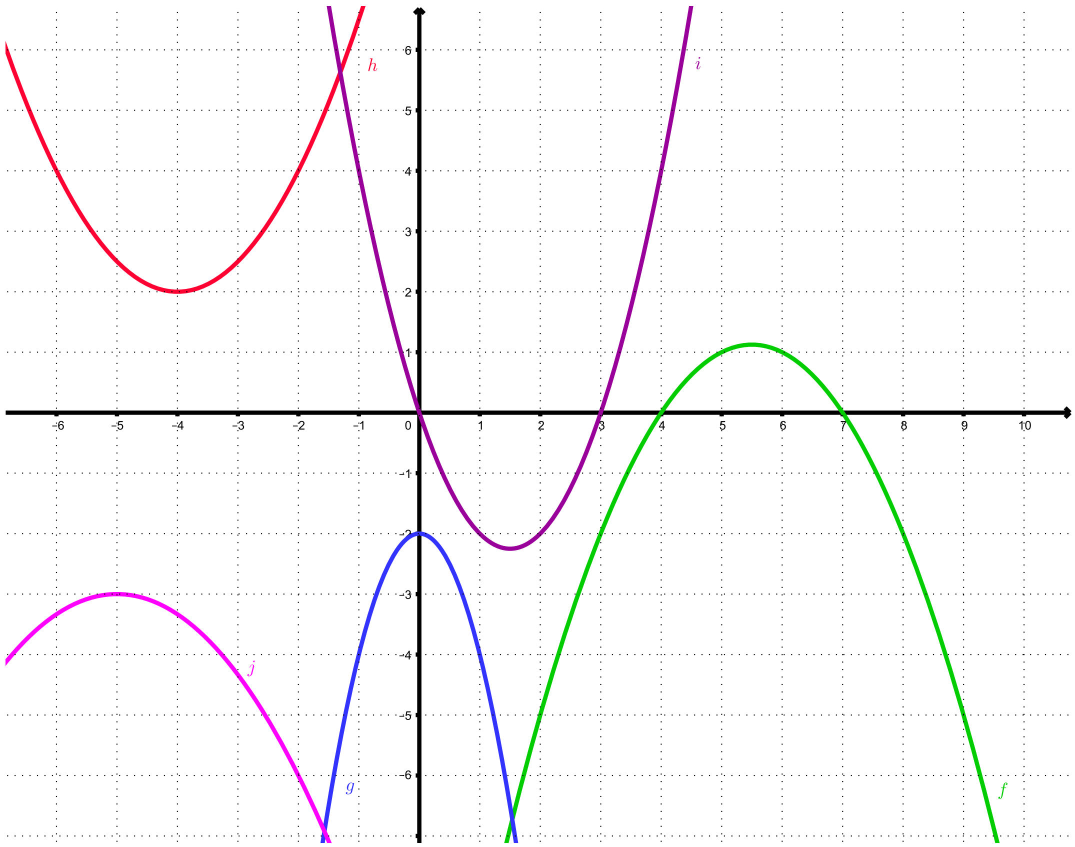
Exercise 6.7. Let $f(x) = \dfrac{1}{2}x^2 + \dfrac{1}{2}x - 6$ be an quadratic function.

a) Compute the coordinates of the intersection points of f with the axes.

b) Compute the coordinates of the parabola's vertex.

c) Compute some other points of the parabola and sketch its graph.

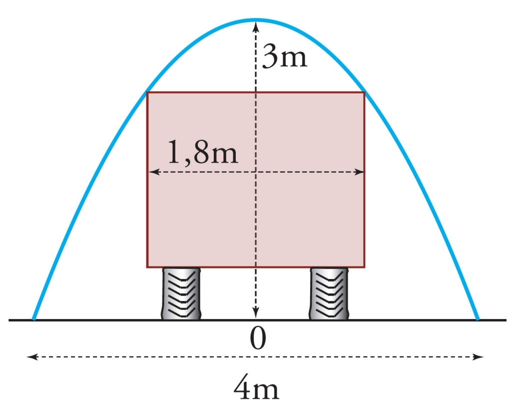
Exercise 6.8. Determine the functional notation of the 5 quadratic functions whose graphs are represented below:



Exercise 6.9. Determine the functional notation of the second degree function f intersecting the x-axis in $x = 5$ and $x = -2$ and passing through the point $A(1; 24)$.

Exercise 6.10. Determine the functional notation of the second degree function f whose vertex is $V(-7; -8)$ and passing through point $A(-3; 7)$.

Exercise 6.11. A tunnel has a parabolic shape with a diameter of 4m and a height of 3m. A truck that is 1,8 m wide wants to drive through this tunnel. What should be its maximum height?



6.5 Intersection of two functions

The method is the same as the one used to calculate the intersection of two lines. All you have to do is solve the system formed by the two parabola equations.

Example. Let's use two functions for this example

$$\mathcal{P} : y = \frac{1}{2}(x+1)^2 \text{ and } d : -x - y = -3.$$

We have to solve this system of two equations:

$$\begin{cases} y &= \frac{1}{2}(x+1)^2 \\ -x - y &= -3 \end{cases}.$$

There are of course several ways to do this, but by isolating y in the equation of the line, we can easily make a comparison (substitution of y).

$$\begin{array}{rcl|l} -x - y &=& -3 & +y+3 \\ y &=& -x+3. & \end{array}$$

Hence, by substituting y, we obtain:

$$\begin{array}{rcl|l} \frac{1}{2}(x+1)^2 &=& -x+3 & \\ \frac{1}{2}(x^2+2x+1) &=& -x+3 & \cdot 2 \\ x^2+2x+1 &=& -2x+6 & +2x-6 \\ x^2+4x-5 &=& 0 & \textit{Factorize} \\ (x+5)(x-1) &=& 0 & \end{array}$$

We have:

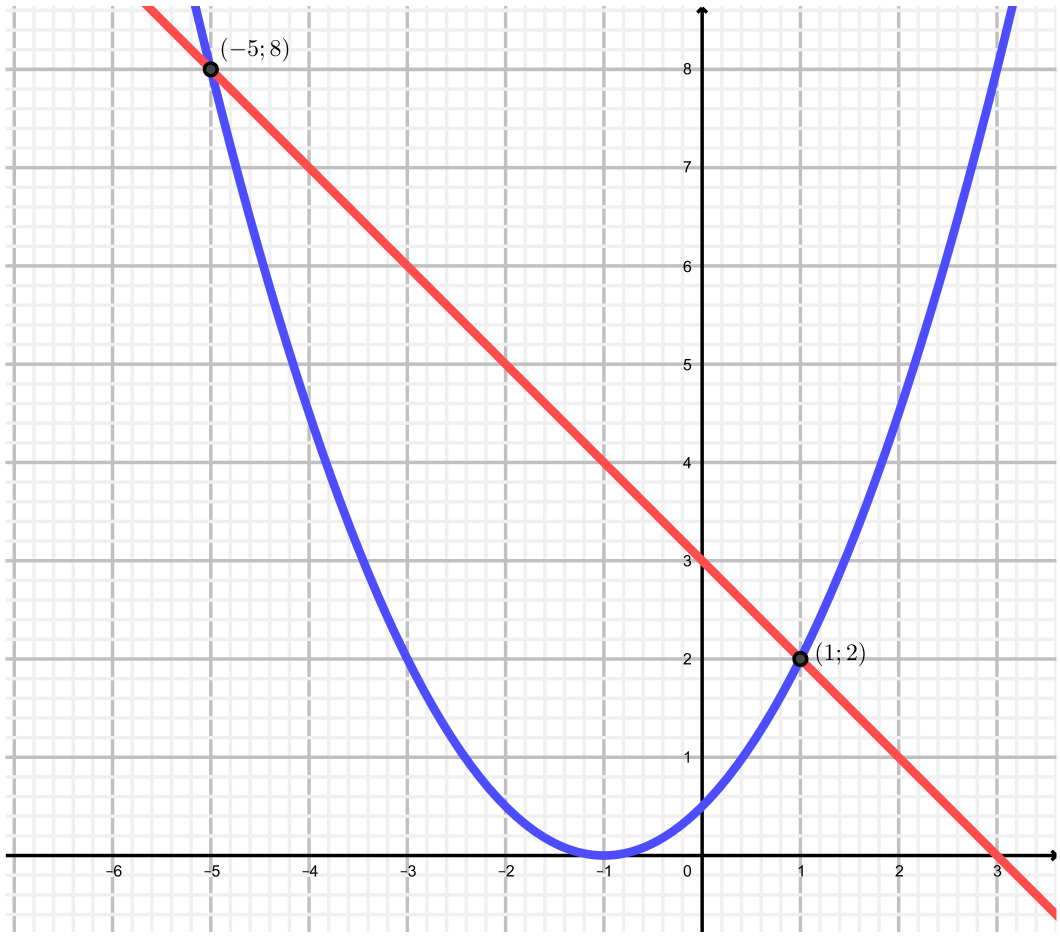
$$x_1 = -5 \text{ and } x_2 = 1.$$

This is the x-coordinate of the two intersection points, which are therefore of the form $(-5; ?)$ and $(1; ?)$.

To find their y-coordinate, we just replace x by what we found in one of the two functional notation.

$y = -(-5) + 3 = 8$. So the point is $(-5; 8)$.

$y = -1 + 3 = 2$. So the point is $(1; 2)$.



Exercise 6.12. Determine the coordinates of the possible intersection point(s) of the parabola $\mathcal{P}$ and the line d in each of the following cases.

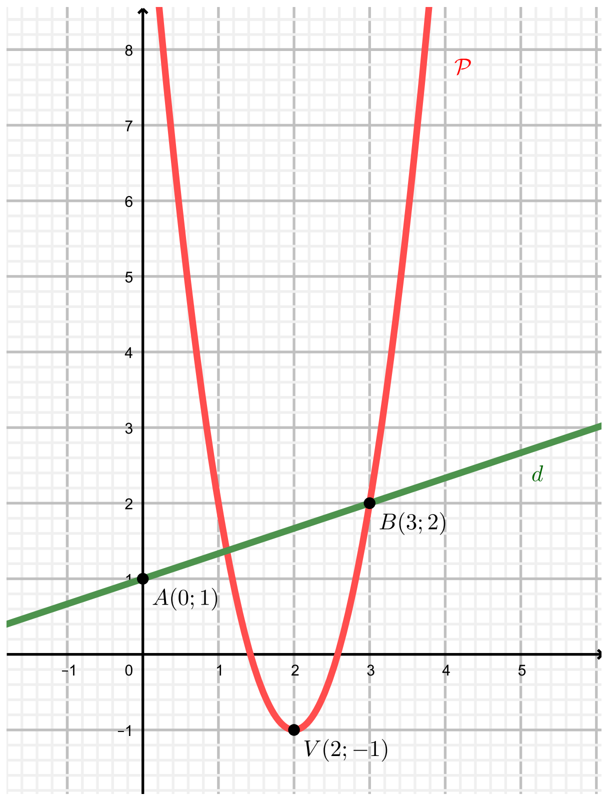
a) $\mathcal{P} : y = x^2 + 7x - 5$ and $d : y = 4x + 5$.
b) $\mathcal{P} : y = 4x^2 - x + 2$ and $d : y = 3x + 1$.
c) $\mathcal{P} : y = 7x^2 + 3x + 2$ and $d : y = x - 2$.

Exercise 6.13. Determine the coordinates of the possible point(s) of intersection of the parabolas $\mathcal{P}_1$ and $\mathcal{P}_2$.

$$\mathcal{P}_1 : y = x^2 + 5x - 2;$$

$$\mathcal{P}_2 : y = 2x^2 + 11x - 9.$$

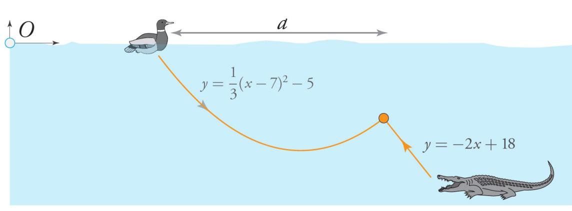
Exercise 6.14. Let's consider this graph.



a) Determine the equation of the parabola $\mathcal{P}$ and of the line d.

b) Compute the intersection points of $\mathcal{P}$ and d.

Exercise 6.15. A duck swimming across a small lake decides to dive in search of food. The situation is illustrated in the picture below, where distances are expressed in meters.



The trajectory of the duck during its dive is given by the quadratic function indicated on the picture. Lurking at the bottom of the lake stands a smart crocodile. The crocodile swims in a straight line towards the duck, after the duck has started to return to the surface. The crocodile's trajectory is also shown in the picture.

a) How far from the bank (origin O) is the duck initially located?

b) At what maximum depth does the duck dive?

c) What's the distance a?(distance from the surface of the lake, between the duck's initial position and the place of its tragic end).

Exercise 6.16 (Exam 2005). A parabola $\mathcal{P}$ is given by its equation

$$y = -2x^2 - 4x + 6.$$

a) Asnwer the following questions and show all the details.

   a) Determine the coordinates of the vertex of $\mathcal{P}$.
   b) Compute the coordinates of the intersection points of $\mathcal{P}$ with the axes.
   c) Sketch the parabola (1 unit = 2 squares).

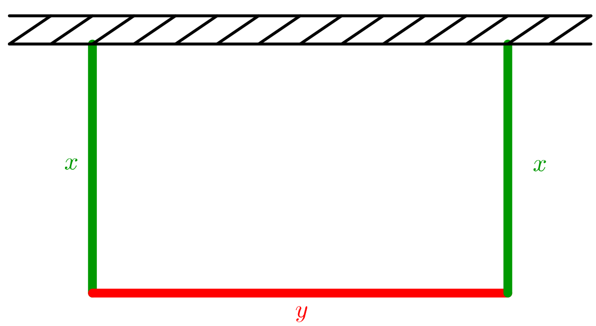
b) By graphical reading, answer the following questions.

   a) How many solutions does the equation $-2x^2 - 4x + 6 = 100$ have?
   b) How many solutions does the equation $-2x^2 - 4x + 6 = -100$ have?
   c) How many solutions does the equation $-2x^2 - 4x + 6 = 8$ have?
   d) How many solutions does the equation $-2x^2 - 4x + 6 = x$ have?

6.6 Optimization

Example. Fences with a total length of 100 metres are available to build a rectangular enclosed area along a straight wall. How large should the enclosure be so that the field it borders has a maximum area?

Let's define x and y as shown in the figure below.



Note that x and y cannot take any value. For example, if $x = 40$ m, then y has to be equal to 20 m since we have a total of 100 m of fence.

Hence, the following constraint is obtained:

Constraint:

$$\begin{aligned} 2x + y &= 100 \\ y &= 100 - 2x. \end{aligned}$$

The area of the enclosure is given by

$$A = x \cdot y = x \cdot (100 - 2x) = 100x - 2x^2.$$

So, the function $A(x) = -2x^2 + 100x$ determine the enclosure's area as a function of the length of the side x.

The abscissa of its vertex, which will be a maximum since $a = -2 < 0$ is given by

$$x_V = \frac{-100}{2 \cdot (-2)} = 25.$$

Hence, the enclosure we are looking for will have the following dimensions $x = 25$ m and $y = 100 - 2 \cdot 25 = 50$ m. Its area is given by

$$A(25) = 25 \cdot 50 = 1250 \text{ m}^2.$$

Example. A real estate company owns 160 apartments which are all rented when it costs 1'000 francs per month. The company estimates that for every increase of 100 francs, ten apartments are vacated. What is the company's current monthly income? What would the rent have to be for the company to have a maximum monthly income?

At the moment, the company's income is given by

$$R = 160 \cdot 1'000 = 160'000 \text{ francs.}$$

Let's define x the number of rent increases of 100 francs.

The table below provides some help.

Number of rent increases	Rent per apartments	Number of apartments	Company's income
0	1'000	160	$1'000 \cdot 160 = 160'000$
1	$1'000 + 100 \cdot 1 = 1'100$	$160 - 10 \cdot 1 = 150$	$1'100 \cdot 150 = 165'000$
2	$1'000 + 100 \cdot 2 = 1'200$	$160 - 10 \cdot 2 = 140$	$1'200 \cdot 140 = 168'000$
3	$1'000 + 100 \cdot 3 = 1'300$	$160 - 10 \cdot 3 = 130$	$1'300 \cdot 130 = 169'000$
x	$1'000 + 100 \cdot x$	$160 - 10 \cdot x$	$(1'000 + 100x) \cdot (160 - 10x)$

The income $R(x)$ is given by

$$\begin{aligned} R(x) &= (1'000 + 100x) \cdot (160 - 10x) \\ &= 160'000 - 10'000x + 16'000x - 1'000x^2 \\ &= -1'000x^2 + 6'000x + 160'000. \end{aligned}$$

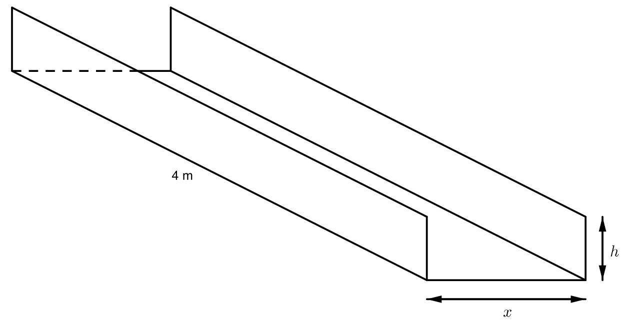
To find the highest income of the company, simply determine the first coordinate of the vertex:

$$x_V = \frac{-b}{2a} = \frac{-6'000}{2 \cdot (-1'000)} = 3.$$

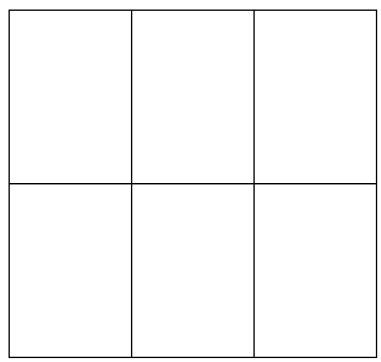
Hence, reaching the highest income corresponds to increase 3 times of 100 francs, in other words, the rent should be 1'300 francs.

Exercise 6.17. Among all the rectangles of perimeter 10, which is the one with the largest area? What is that area?

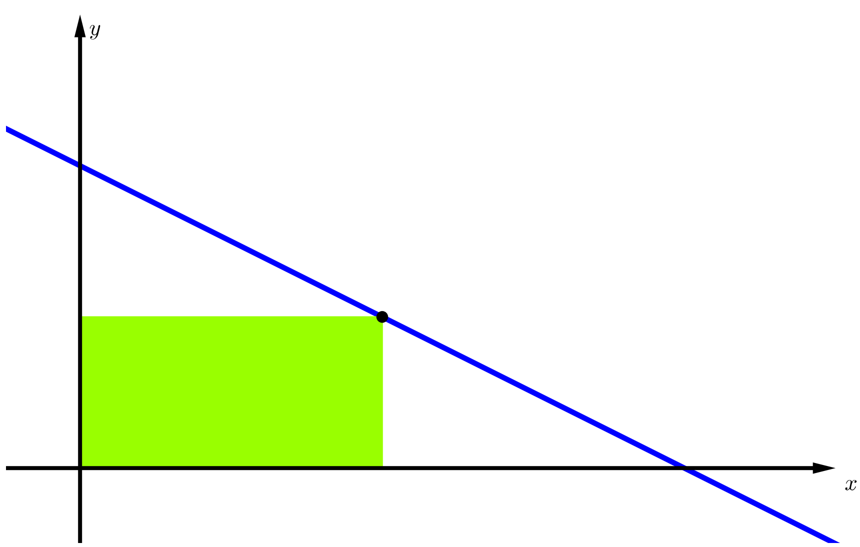
Exercise 6.18. A rectangular metal plate which is 4 meters long and 40 centimeters wide is bent to form a rectangular parallelepiped gutter. What dimensions should this gutter have in order to have a maximum volume?



Exercise 6.19. We have 288 meters of wire fencing to build 6 identical enclosures for a zoo according to the plan below. How large should these enclosures be to maximize their surface?



Exercise 6.20. Let's consider the colored rectangle below, limited by the x-axis, the y-axis and the line of equation $y = -\frac{1}{2}x + 3$. What are the dimensions of this rectangle if we want its area to be maximal?



Exercise 6.21. A real estate company owns 180 studios, all of which are rented for 300 francs a month. The company estimates that for every 10 franc increase in rent, 5 studios are vacated. What should the rent be to ensure that the company has a maximum monthly income?

Exercise 6.22. When tickets for a football match cost 40 francs, there are 14'000 spectators. Each increase of 1 franc means a loss of 280 spectators. How much should the admission price be set to maximise the club's income?

Exercise 6.23. A car park with a total capacity of 600 spots charges 120 francs per month per parking spot. Currently, 480 spaces are occupied. If the monthly price was reduced by 1 franc, then 5 additional spaces would be occupied. What should the monthly price be in order to maximise income?

Exercise 6.24. Let $d : y = -x + 1$ be a line and $p : y = x^2 + 2x - 3$ be a parabola.

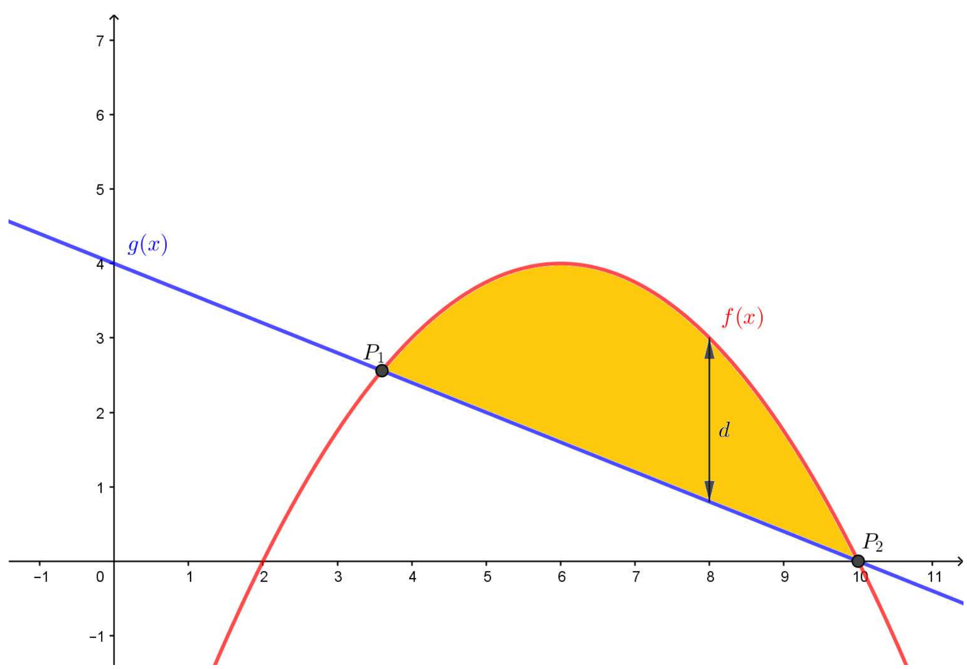
a) Determine the coordinates of the intersection points $I_1(x_1; y_1)$ and $I_2(x_2; y_2)$ of d and p.

b) Determine the maximal vertical distance between the parabola and the line in the interval $[x_1; x_2]$.

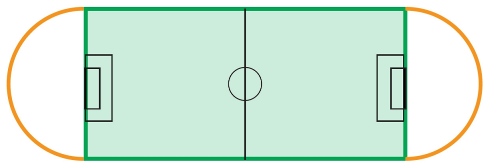
Exercise 6.25. We have the parabola given by $f(x) = -0,25x^2 + 3x - 5$ and the equation of a line $g(x) = -0,4x + 4$.

a) Determine the coordinates of the intersection points P_1 and P_2 of the line and the parabola.

b) Determine the greatest distance d between the parabola and the line in the shaded area.



Exercise 6.26. A 400 m running track is formed by two half-discs at its ends, as shown below. Determine the length and width of the football field in order to get the maximum area.
Reminder: The perimeter P of a circle is given by $P = 2\pi r$.



6.7 Application to economics

This section is a continuation of the section in chapter 5. However, we will consider an additional variable. Indeed, we know that the number of objects sold depends strongly on their price. The more expensive an item is, the less demand there will be, and vice versa. There is therefore a direct relationship between the number of objects x, i.e. the demand and the selling price p :

$$x = m - np.$$

Here, the value m describes the maximum number of items that can be provided, while the value n represents the quantity sold decrease for each price increase of 1 franc. Thanks to this relationship, it will now be possible to express costs, income and profit directly as a function of the price p and not as a function to the number of objects x.

Example. A factory produces toasters. A study established that the demand x based on the price p is given by the relation $x = 10'200 - 300p$. The company calculates that the production will require a fixed investment of 14'400 francs to which will be added 8 francs per toaster manufactured. What price will the company have to set for its toasters if it wants to make a maximum profit?

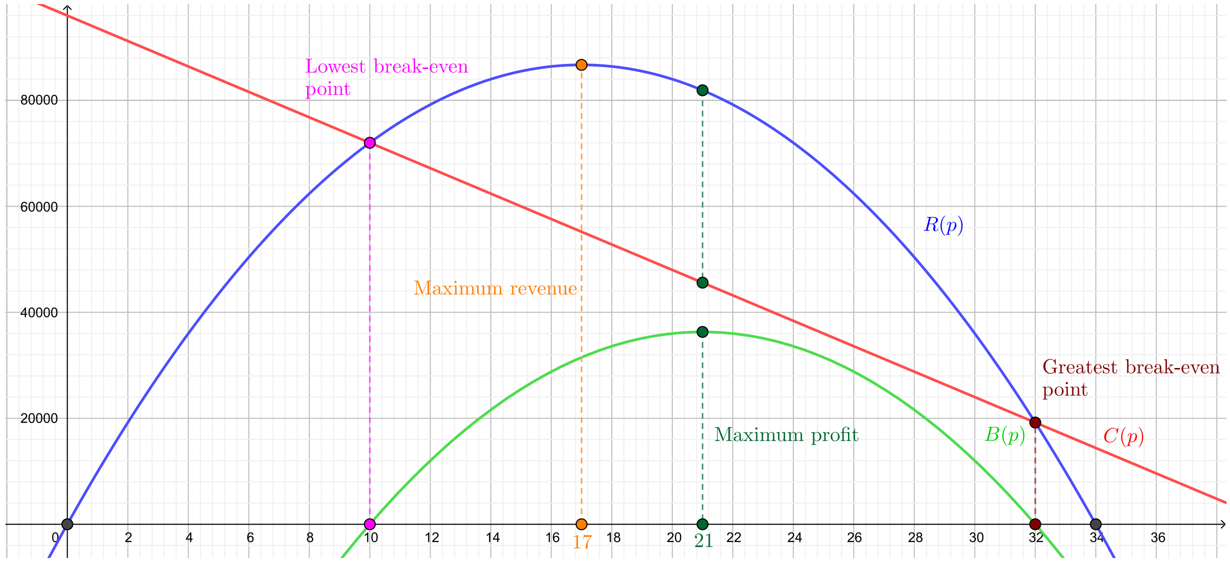
The first step is to determine the economic functions:

$$\begin{array}{lll}
\textbf{Demand:} & x & = 10'200 - 300p. \\
\textbf{Revenue:} & R(p) & = px = p(10'200 - 300p) = 10'200p - 300p^2. \\
\textbf{Costs:} & C(p) & = 14'400 + 8x = 14'400 + 8(10'200 - 300p) = -2'400p + 96'000. \\
\textbf{Profit:} & P(p) & = R(p) - C(p) = 10'200p - 300p^2 - (-2'400p + 96'000) \\
 & & = -300p^2 + 12'600p - 96'000.
\end{array}$$

By putting $R(p) = 0$, we determine that the revenue is zero for selling prices of $p = 0$ and $p = 34$. For a price between these two values, the revenue will be positive. The revenue will be maximal at the selling price $p = \dfrac{0 + 34}{2} = 17$ francs.

By solving the equation $P(p) = 0$, or equivalently $R(p) = C(p)$, we find the break-even points $p = 10$ francs and $p = 32$ francs.

At the selling price $p = \dfrac{10 + 32}{2} = 21$ francs, we will have a maximum profit of $P(21) = 36'300$ francs.



Exercise 6.27. A market study established the following relationship between the price p of a calculator and the volume x of calculators sold:

$$x = 3'920 - 28p.$$

The production cost of x calculators (in francs) is given by

$$C(x) = 30x + 11'872.$$

.

a) Determine the revenue $R(p)$ as a function of the selling price p of the calculators.

b) Determine the production costs $C(p)$ as a function of the selling price p of the calculators.

c) Determine the maximal profit.

Exercise 6.28. A mathematics book author wants to sell his book. A market study tells him that the demand is measured by $x = 1'200 - 15p$ where p is the selling price of the book. The production costs are estimated at 9'300 francs of fixed costs and 8 francs of variable costs per book. At what selling price should he set his book if he wishes

a) A maximal revenue? What's the value of the revenue?

b) To reach the lowest break-even point?

c) To reach the greatest break-even point?

d) A maximal profit? What's the value of the profit?

Exercise 6.29. One wishes to put on the market a new product whose demand (number of items sold) is given by

$$x = 660 - 22p$$

with p the selling price. The production costs are 820 francs plus 7 francs of variable costs per object.

a) Determine the functions which gives the revenue, the costs and the profit as a function of the price.

b) Determine the range of price which allows a positive revenue.

c) Compute the price and the number of objects that maximise the revenue. What will be this revenue?

d) Determine the range of price which allows a positive profit.

e) Compute the price and the number of objects that maximise the profit. What will be this profit?

Exercise 6.30. The SOSCREWED company launches a new lock on the market. A study of the demand and its production costs showed that its monthly profit P based on the selling price x is given by

$$P(x) = -210x^2 + 4'620x - 13'650.$$

.

a) What will be its monthly profit if the selling price is set at 8 francs?

b) What should the selling price be for a zero profit?

c) What should the selling price be for a maximal profit?

d) What will be this profit?

e) 20 francs for a lock, is it a good idea?

f) In what interval will the profit increase?

Exercise 6.31. A young inventor plans to commercialize an electronic parking disc. After an initial investment in equipment of 72'000 francs, it would cost him 12 francs per item to manufacture. According to a market study, the demand x for the product based on the price p would be given by the relation

$$x = 42'000 - 1'400p.$$

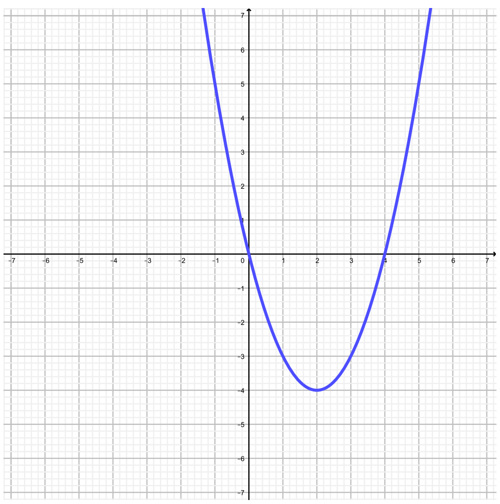
a) Determine, under these conditions, the unit price that will bring the greatest profit, the amount of the maximum profit and the number of items sold.

b) This young inventor thinks it would be good to set the unit price above the one found at point a). He then makes a profit of 35'800 francs. What selling price did he set?

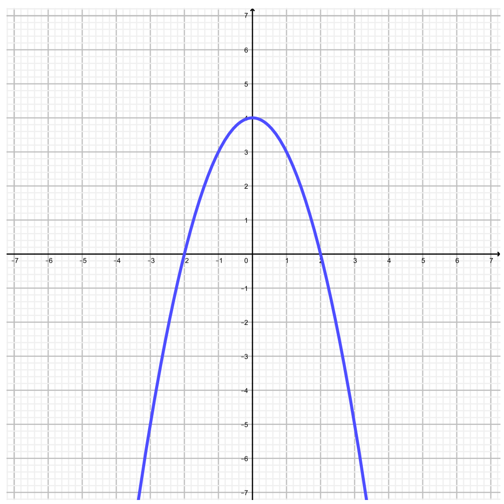
6.8 Solutions

Exercise 6.1.

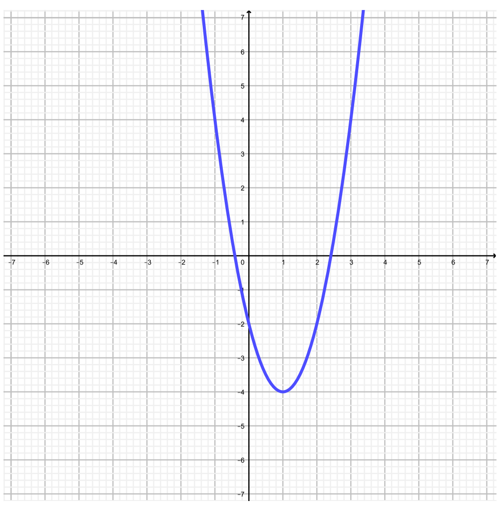
a)



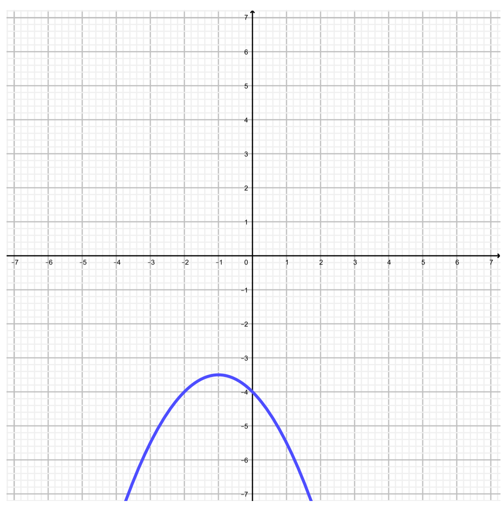
b)



c)



d)



Exercise 6.2.

a) Yes because $f(1) = 6$.

b) No because $f(2) = 9$.

c) $y = -9$ because $f(3) = -9$.

Exercise 6.3.

a) $I_1(-1; 0), I_2(3; 0), I_3(0; -3)$ and $V(1; -4)$, minimum

b) $I_1(0; 0), I_2(5; 0)$ and $V\left(\frac{5}{2}; \frac{25}{4}\right)$, maximum

c) $I(0; 3)$ and $V(0; 3)$, minimum

d) $I(0; -4)$ and $V\left(\frac{5}{4}; -\frac{7}{8}\right)$, maximum

Exercise 6.4.

a) 5 m.

b) 3 m.

c) 7 m.

d) 3, 4 m.

Exercise 6.5.

a) $V\left(4;-3\right)$, minimum

b) $V\left(1;2\right)$, minimum

c) $V\left(2;0\right)$, minimum

d) $V\left(-3;0\right)$, maximum

e) $V\left(0;-1\right)$, minimum

f) $V\left(\frac{1}{2};-\frac{245}{4}\right)$, minimum

g) $V\left(\frac{1}{2};\frac{49}{2}\right)$, maximum

h) $V\left(10;0\right)$, maximum

i) $V\left(5;\frac{25}{2}\right)$, maximum

j) $V\left(-\frac{5}{2};-\frac{75}{4}\right)$, minimum

k) $V\left(0;-5\right)$, minimum

l) $V\left(0;7\right)$, minimum

Exercise 6.6.

a) $I_1(3;0), I_2(4;0)$ and $I_3(0;24)$

b) $I_1(-1;0), I_2(3;0)$ and $I_3(0;-6)$

c) $I_1(5;0), I_2(-3;0)$ and $I_3(0;5)$

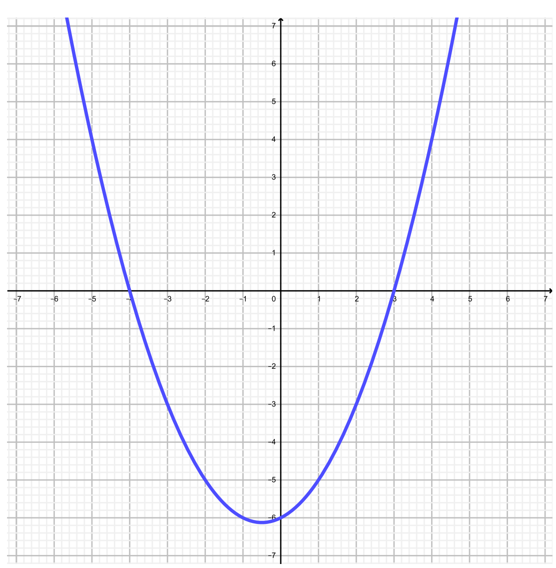
d) $I(0;-18)$

Exercise 6.7.

a) $I_1(-4;0)$, $I_2(3;0)$ and $I_3(0;-6)$.

b) $V\left(-\frac{1}{2};-\frac{49}{8}\right)$.

c)



Exercise 6.8.

$$f(x) = -\frac{1}{2}(x-4)(x-7) \quad \text{(Polynomial form: } f(x) = -\frac{1}{2}x^2 + \frac{11}{2}x - 14\text{)}.$$
$$g(x) = -2x^2 - 2.$$
$$h(x) = \frac{1}{2}(x+4)^2 + 2 \quad \text{(Polynomial form: } h(x) = \frac{1}{2}x^2 + 4x + 10\text{)}.$$
$$i(x) = x(x-3) \quad \text{(Polynomial form: } i(x) = x^2 - 3x\text{)}.$$
$$j(x) = -\frac{1}{3}(x+5)^2 - 3 \quad \text{(Polynomial form: } j(x) = -\frac{1}{3}x^2 - \frac{10}{3}x - \frac{34}{3}\text{)}.$$

Exercise 6.9. $f(x) = -2(x-5)(x+2)$.

Exercise 6.10. $f(x) = \frac{15}{16}(x+7)^2 - 8$.

Exercise 6.11. $2,39$ m.

Exercise 6.12.

a) $I_1(2;13)$ and $I_2(-5;-15)$.

b) $I\left(\frac{1}{2};\frac{5}{2}\right)$.

c) No intersection point.

Exercise 6.13. $I_1(-7;12)$ and $I_2(1;4)$.

Exercise 6.14.

a) $\mathcal{P} : y = 3(x-2)^2 - 1$ and $d : y = \frac{1}{3}x + 1$.

b) $I_1\left(\frac{10}{9};\frac{37}{27}\right)$ and $I_2\,(3;2)$.

Exercise 6.15.

a) $3,13$ m.

b) 5 m.

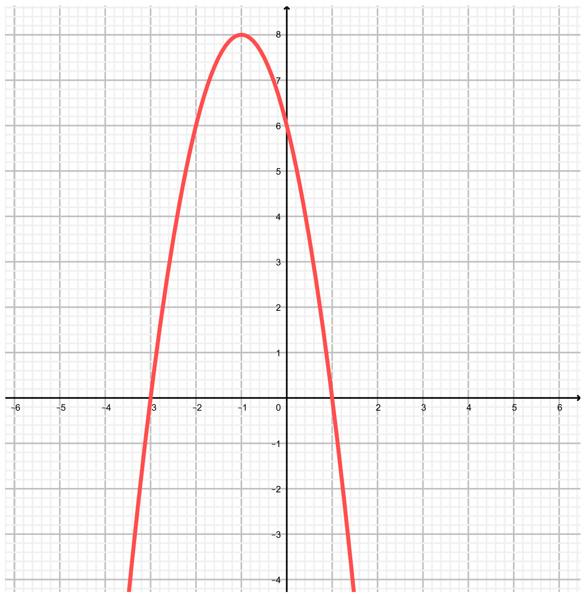
c) $a \cong 6,87$ m.

Exercise 6.16.

a) a) $V(-1;8)$.

   b) $I_1(-3;0)$, $I_2(1;0)$ and $I_3(0;6)$.

   c)



b) a) 0 solution.

   b) 2 solutions.

   c) 1 solution.

   d) 2 solutions.

Exercise 6.17. A square whose length is $2,5$ and area $6,25$.

Exercise 6.18. $x = 20$ cm and $h = 10$ cm.

Exercise 6.19. 16 m x 18 m.

Exercise 6.20. 3 and $\dfrac{3}{2}$.

Exercise 6.21. 330 francs per month.

Exercise 6.22. The admission price should be 45 francs.

Exercise 6.23. 108 francs.

Exercise 6.24.

a) $I_1(-4;5)$ and $I_2(1;0)$.

b) $\dfrac{25}{4}$.

Exercise 6.25.

a) $(3,6;2,56)$ and $(10;0)$.

b) $2,56$.

Exercise 6.26. 100 m and $\dfrac{200}{\pi} \cong 63,66$ m.

Exercise 6.27.

a) $R(p) = 3'920p - 28p^2$.

b) $C(p) = 129'472 - 840p$.

c) 72'828 francs.

Exercise 6.28.

a) 40 francs, 24'000 francs.

b) 18 francs.

c) 70 francs.

d) 44 francs, 10'140 francs of profit.

Exercise 6.29.

a) $R(p) = 660p - 22p^2$, $C(p) = 5'440 - 154p$ and $P(p) = -22p^2 + 814p - 5'440$.

b) Ranging from 0 to 30 francs.

c) 330 objects at the price of 15 francs. The income will then be 4'950 francs.

d) Ranging from $8,75$ francs to $28,25$ francs.

e) 253 objects at the price of $18,5$ francs. The profit will then be $2'089,5$ francs.

Exercise 6.30.

a) 9'870 francs.

b) For $3,50$ francs and for $18,50$ francs.

c) 11 francs.

d) 11'760 francs.

e) Definitely not, the profit would be negative.

f) Between 3,50 francs and 11 francs.

Exercise 6.31.

a) The unit price that will bring the greatest benefit is 21 francs, the maximum profit is 41'400 francs and 12'600 items will be sold in this way.

b) The price was set at 23 francs.

6.9 Chapter objectives

At the end of this chapter, the student should be able to

6.1 ☐ Sketch the graph of a quadratic function.

6.2 ☐ Compute the coordinates of the intersection points of a parabola with the axes.

6.3 ☐ Compute the coordinates of the vertex of a parabola and determine whether it's a minimum or maximum.

6.4 ☐ Determine the equation of a parabola (polynomial, standard or factored form) from its graph, from its vertex and a point or from a point and its intersection points with the x-axis.

6.5 ☐ Compute the intersection point(s) of a parabola and another parabola (or a line).

6.6 ☐ Solve an optimization problem using quadratic functions.

6.7 ☐ Solve a problem related to economics using quadratic functions

Chapter 7

Exponential functions and logarithms

7.1 Introduction

In the previous chapters, we studied functions whose functional notations was of the form

$$y = \text{variable}^{\text{constant}},$$

like $f(x) = x^2$ or $f(x) = 2x + 3$.

These functions studied so far help to model fairly simple phenomena such as uniform motion in physics or simple interest in economics. More sophisticated processes, such as compound interest (finance), radioactivity (carbon-14 dating, radioactive waste, ...) or population evolution (human, bacteriological, virus propagation, prey-predators, ...) to mention only a few examples, require functions of a different kind: *i.e. exponential and logarithmic functions.* In addition to this practical aspect, exponential and logarithmic functions have remarkable mathematical properties.

Example. A 0,1 mm thick sheet of paper is folded in half, then folded 4 times, then folded 8 times, and so on. Would it be possible to achieve a thickness greater than :

2 m, 20 m , 1 km , the distance from Earth to the sun?

On one hand, the extreme thinness of 0,1 mm makes you doubt that you can exceed heights such as 20 m, 1 km, and even more the Earth-Sun distance; by doubling something very small, you certainly get something very small! But, by doubling it many times you end up exceeding any number. To get an idea, let's calculate the thicknesses obtained after the first folds.

Number of folds	Thickness in mm
0	$0,1$
1	$0,2$
2	$0,4$
3	$0,8$
4	$1,6$
5	$3,2$
6	$6,4$
7	$12,8$
8	$25,6$
9	$51,2$
10	$102,4$

We can see that after 10 folds the thickness is about 10 cm, which is not very high yet. After 15 folds, the thickness is about 3,2 m, which is already more surprising. After 20 folds, it is of the order of 100 m and finally after 60 folds, it is equal to $0,1 \cdot 2^{60}$ mm. However,

$$0,1 \cdot 2^{60} = 1,152922 \cdot 10^{17}.$$

it's a number of millimeters. Since 1 km $= 10^6$ mm, that makes

$$115'292'200'000 \text{ km}.$$

By comparison, the distance from the Earth to the Sun is about

$$149'597'870'700 \text{ km}.$$

Example. Sometimes you get a message in your mailbox that says "When you receive this letter, send me 10 francs then copy the letter ten times and send it to ten of your acquaintances. So you will receive 100 francs after giving only 10 francs. Please do not interrupt this chain."

What to think of such a thing?

Let's suppose that everyone plays the game. Initially, 10 letters are sent, then, 10^2 =100, at the third round, 10^3 =1000, etc. So, after only 10 rounds, the number of letters written at this stage (and the number of people) reaches

$$10^{10} = 10'000'000'000,$$

which is more than the world's population! So it's a dead end. A lot of people will have lost 10 francs, which means that there will be no one left after them to carry on. It is understandable that the law forbids this kind of practice: the first in the chain simply steal from the next ones.

Example. A legend, probably apocryphal, tells that Sissa who invented chess was summoned by his master, king of Persia: "Your game has given me back the joy of life! I offer you what you desire!"

The wise man wanted nothing and said nothing. The offended king became angry: "Speak up, you insolent man! Are you afraid I cannot grant your wishes?"

The wise man was hurt by this tone and decided to take revenge: "I accept your gift. You will place a grain of rice on the first square of the chessboard."

"And that's it? Are you making fun of me?"

"Not at all, my lord. Then you will have two grains put on the second square, four on the third and so on..."

The king was really angry: "Since you honour my generosity so badly, go away! Your bag of rice will be brought to you tomorrow and don't bother me anymore!"

The next morning, the king was awakened by his panicked bursar: "My lord, it's a disaster! We can't deliver the rice! Our mathematicians worked all night. There's not enough rice in the whole kingdom to grant the scientist's wish."

Indeed, just on the last square alone we would need $2^{63} = 9'223'372'036'854'775'808$ grains of rice.

If he wanted to provide the total quantity of rice, the king would have to accumulate all the harvests made on Earth for 5000 years! If his silo measures 4 metres by 10 metres, its height would have to be 300 million kilometres!

7.1.1 Exponential functions

Definition. An *exponential function* is a function of the form

$$y = \text{constant base}^{\text{variable}},$$

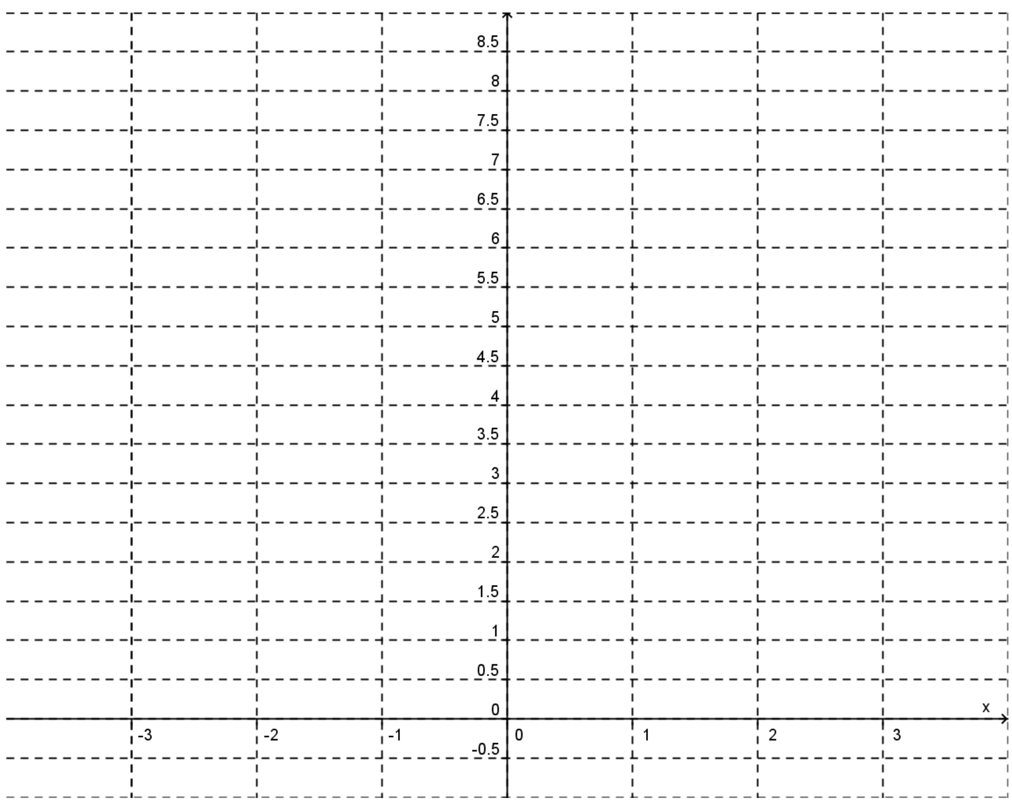
with the constant base strictly positive and different from 1.

Example. $f(x) = 2^x$ or $f(x) = 0,5^x$.
We say that $f(x) = 2^x$ is an *exponential function with base 2.*

Example. Let $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$ be exponential functions.
Let's fill in these table of values in order to sketch functions f and g.

x	-3	-2	-1	0	1	2	3
$f(x)$							

x	-3	-2	-1	0	1	2	3
$g(x)$							



Exercise 7.1. Sketch the following functions:

a) $f(x) = 5^x$ b) $f(x) = \left(\dfrac{1}{5}\right)^x$

c) $f(x) = 10^x$ d) $f(x) = \left(\dfrac{1}{10}\right)^x$

For which values of a is the function $f(x) = a^x$ increasing? Decreasing?

7.2 Simple exponential equations

First of all, let's remind ourselves of the properties of the exponents:

1. $a^0 = 1$.
2. $a^1 = a$.
3. $a^m \cdot a^n = a^{m+n}$.
4. $\dfrac{a^m}{a^n} = a^{m-n}$.
5. $a^{-n} = \dfrac{1}{a^n}$.
6. $(a^n)^m = a^{n\cdot m}$.
7. $(a \cdot b)^n = a^n \cdot b^n$.
8. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$.

We will begin by discussing two specific cases.

Example (same basis). Solve the equation $5^{2x-3} = 5^{x+7}$.

$$\begin{array}{rcl|l} 5^{2x-3} & = & 5^{x+7} & \\ 2x-3 & = & x+7 & -x+3 \\ x & = & 10. & \end{array}$$

Example (connected basis). Solve the equation $3^{5x-8} = 9^{x+2}$.

$$\begin{array}{rcl|l} 3^{5x-8} & = & 9^{x+2} & \text{We express with the same basis} \\ 3^{5x-8} & = & (3^2)^{x+2} & \text{Property of an exponent of another exponent} \\ 3^{5x-8} & = & 3^{2(x+2)} & \\ 3^{5x-8} & = & 3^{2x+4} & \\ 5x-8 & = & 2x+4 & -2x+8 \\ 3x & = & 12 & :3 \\ x & = & 4. & \end{array}$$

Exercise 7.2. Solve the following equations.

a) $7^{x+6} = 7^{3x-4}$	b) $6^{7-x} = 6^{2x+1}$
c) $3^{2x+3} = 3^{(x^2)}$	d) $2^x = 8$
e) $3^{4x} = 9^{x+5}$	f) $3^{5x-8} = 9^{x+2}$
g) $9^{(x^2)} = 3^{3x+2}$	h) $27^{x-1} = 9^{2x-3}$
i) $4^{x-3} = 8^{4-x}$	j) $5^{x+2} \cdot 25^{-x} = 625$
k) $\left(\dfrac{1}{2}\right)^{6-x} = 2$	l) $2^{-100x} = 0,5^{x-4}$

7.3 Compound interest formula

Exponential functions are widely used when describing the evolution of different populations (animals, bacteria, etc.), the evolution of a price (house, car, etc.) or for the calculation of compound interest.

The last is a perfect illustration of exponential growth.

If we put 1'000 francs in a bank account with an annual interest rate of 9%, after one year we will have the amount of

$$P_1 = 1'000 + 1'000 \cdot 0,09 = 1'090 \text{ francs.}$$

The following year, we will not have an increase of 90 francs, because we calculate the 9% on the existing amount, i.e. 1'090 francs.

Hence,

$$P_2 = P_1 + P_1 \cdot 9\% = 1'090 + 1'090 \cdot 0,09 = 1'188,10 \text{ francs.}$$

In general, if we place a principal P_0 on a bank account at a rate of r (in decimal code), we can calculate the different amounts as follows:

$$\begin{array}{rcl} P_1 & = & P_0 + P_0 \cdot r = P_0 \cdot (1+r) \\ P_2 & = & P_1 + P_1 \cdot r = P_1 \cdot (1+r) = P_0 \cdot (1+r) \cdot (1+r) = P_0 \cdot (1+r)^2 \\ \vdots & \vdots & \vdots \\ P_t & = & P_0 \cdot (1+r)^t. \end{array}$$

Here, t represents the time period, which can be expressed in hours, days, weeks, months, years, etc. depending on the context.

Theorem. *The compound interest formula is*

$$\boxed{P_t = P_0 \cdot (1+r)^t.}$$

where

$$\begin{array}{rcl} P_t & = & \textit{Final amount (after t years, t days etc.).} \\ r & = & \textit{Interest rate (WARNING, write in decimal code !).} \\ t & = & \textit{Number of years, days, etc. .} \\ P_0 & = & \textit{Principal (original amount invested).} \end{array}$$

Remark. In case of amortization or if the value decreases over time, r is negative.

Example.

1. We invest 1'000 francs in a bank account at a rate of 3.5% per year. How much will the amount be in seven years?

   We know that $P_0 = 1'000$, $r = 0,035$ and $t = 7$. We're looking for P_7.

$$P_7 = 1'000(1 + 0,035)^7 \cong 1'272,30 \text{ francs.}$$

2. The purchase of a car is amortized at an annual rate of r. Knowing that its initial price was 40'000 francs, determine the rate r so that only 10'000 francs remain to be paid back after 9 years.

   We know that $t = 9$, $P_0 = 40'000$ and $P_9 = 10'000$. We're looking for t.

$$\begin{array}{rcl|l} 10'000 & = & 40'000(1+r)^9 & : 40'000 \\ 0,25 & = & (1+r)^9 & \sqrt[9]{} \\ 0,857 & \cong & 1+r & -1 \\ r & \cong & -0,143. & \end{array}$$

   Donc $r \cong -14,3\%$.

Exercise 7.3. A principal invested at an interest rate of 5% is today worth 1'000 francs.

a) What will be the amount after 6 years?

b) What was the amount 4 years ago?

c) At what rate would it have to be invested for it to double in 11 years?

Exercise 7.4. A population of 350 elks, all of them one year old, is introduced into a nature reserve. Each year, 8% of the population disappears. Give the approximate number of animals surviving after 5, 8 and 12 years.

Exercise 7.5. A disease is observed in a country and we want to study the speed of its spread. There are now 1'000 infected people, and it has been observed that in one month, the number of patients increases by 14% compared to the previous month.

a) Determine the $S(t)$ function that estimates the number of sick people based on the number of months t that have elapsed since today.

b) How many infected people will there be in three months?

c) How many infected people were there two months ago?

Exercise 7.6. A forest grows exponentially. It now occupies $72'515$ m^3. Twelve years ago, it was $47'228,5$ m^3

a) What is, in %, the annual growth rate of this forest?

b) What was its volume five years ago?

c) What will be its volume in 7 years?

7.4 Logarithms

Example. After how many years will 1'000 francs invested at a 2% rate in an account reach the amount of 1'268,24 francs?

To answer such a question, the equation

$$1'268,24 = 1'000(1+0,02)^t.$$

has to be solved.

However, we do not have a way to solve an equation where the unknown is an exponent.

What follows will help us overcome this problem.

Definition. A *logarithmic function* with base a is defined as the reciprocal function of the exponential function with base a. We have the relation

$$\boxed{y = \log_a(x) \Leftrightarrow x = a^y.}$$

Where a is a positive real number different from 1 and x a strictly positive real number.

In other words, the logarithm answer the following question:

$$\boxed{\text{"}a \text{ to the power what gives } x\text{?"}}$$

Example.

1. $\log_2(8) = 3$ because $2^3 = 8$.
2. $\log_3(27) = 3$ because $3^3 = 27$.
3. $\log_4(2) = \frac{1}{2}$ because $4^{\frac{1}{2}} = 2$.
4. $\log_{10}(10) = 1$ because $10^1 = 10$.
5. $\log_{10}\left(\frac{1}{100}\right) = -2$ because $10^{-2} = \frac{1}{100}$.
6. $\log_3(1) = 0$ because $3^0 = 1$.
7. $\log_2(-4)$ doesn't exist. Indeed, it's impossible to reach any negative number with a power of 2.
8. Same for $\log_a(0)$.

Remark.

1. It's impossible to compute the logarithm of zero or of a negative number.
2. When no base is specified, it is considered to be with base 10. In other words, $\log(x) = \log_{10}(x)$.
3. We call *natural logarithm* the logarithm defined with base e. It is written $\ln(x)$. The number e ($e \cong 2,718$) is a mathematical constant called *Euler's number*, after the name of the Swiss mathematician (1707-1783). It is used in many domains such as population evolution or differential and integral calculus.

Exercise 7.7. Compute the following logarithms.

a) $\log_{10}(1'000)$	b) $\log_2(8)$
c) $\log_3(27)$	d) $\log_2(64)$
e) $\log_2(1'024)$	f) $\log_{10}(10^7)$
g) $\log_3(3)$	h) $\log_5(1)$
i) $\log_{10}\left(\frac{1}{10}\right)$	j) $\log_9\left(\frac{1}{81}\right)$
k) $\log_2\left(\frac{1}{8}\right)$	l) $\log_3\left(\sqrt{3}\right)$
m) $\log_{16}(4)$	n) $\log_5\left(\sqrt[9]{5}\right)$
o) $\ln(e^3)$	p) $\ln\left(\frac{1}{e^2}\right)$

Exercise 7.8. Find, if possible, the value of x that satisfies the following equalities:

a) $\log_2(x) = 4$ b) $\log_3(x) = 5$
c) $\log_4(x) = 3$ d) $\log_x(256) = 4$
e) $\log_x(125) = 3$ f) $\log_x(1'000) = 3$

7.5 Logarithmic functions

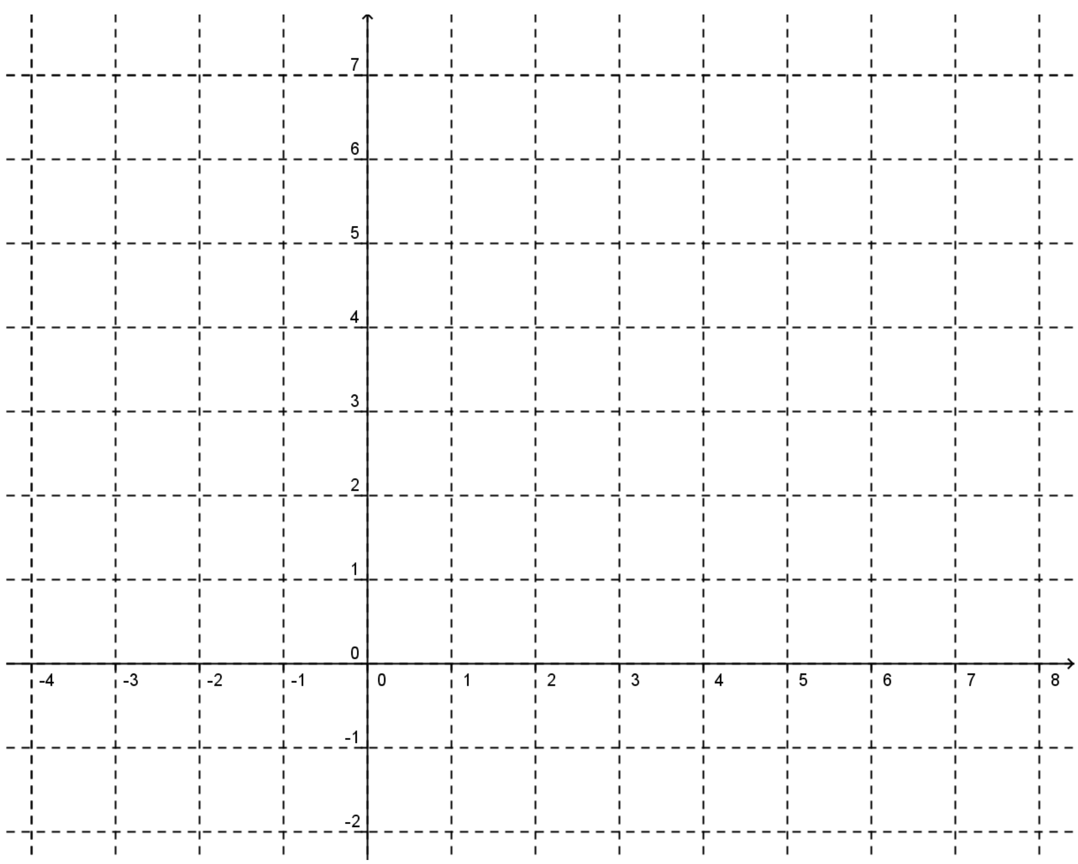
Example. Let's see those two functions:

$$f(x) = 2^x \text{ and } g(x) = \log_2(x).$$

Let's fill in these table of values and sketch the graphs

x	-2	-1	0	1	2	3
$f(x)$						

x	-1	0	$0,25$	$0,5$	1	2	4	8
$g(x)$								



Let's sketch now the line of equation $y = x$.

Since the function $g(x) = \log_2(x)$ is the reciprocal of the function $f(x) = 2^x$, we notice that they are perfectly symmetrical with respect to the line $y = x$.

7.6 Properties of logarithms

Theorem. *With any base a, we have*

1. $\log_a(1) = 0$.
2. $\log_a(xy) = \log_a(x) + \log_a(y)$.
3. $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.
4. $\log_a(x^n) = n \cdot \log_a(x)$

Proof.

1. $\log_a(1) = 0$ because $a^0 = 1$.
2. It exist $u, v \in \mathbb{R}$ such that
$$x = a^u \text{ and } y = a^v.$$
In this situation, we have
$$\begin{aligned} &\log_a(x) = \log_a(a^u) = u; \\ &\log_a(y) = \log_a(a^v) = v. \end{aligned}$$
So,
$$\begin{array}{rcl} \log_a(xy) & = & \log_a(a^u \cdot a^v) \\ & = & \log_a(a^{u+v}) \\ & = & u + v \\ & = & \log_a(x) + \log_a(y) \end{array}$$
3. We have
$$\log_a(x) = \log_a\left(\frac{x}{y} \cdot y\right) = \log_a\left(\frac{x}{y}\right) + \log_a(y).$$
Therefore, we have
$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y).$$
4. Let's define $m = \log_a(x)$.
In exponential form, we have
$$\left.\begin{array}{rcl} x & = & a^m \\ x^n & = & (a^m)^n \\ x^n & = & a^{mn}. \end{array}\right| (...)^n$$
If we come back with the logarithmic form, we get
$$\log_a(x^n) = mn.$$
Since $m = \log_a(x)$, we have
$$\log_a(x^n) = n \cdot \log_a(x).$$

$\square$

Exercise 7.9. Compute the following expressions using the proporties of logarithms

a) $\log_5(100) - \log_5(4)$

b) $\log_{10}(6) - \log_{10}(6'000)$

c) $\log_3\left(\dfrac{9}{2}\right) + \log_3(6)$

d) $2 \cdot \log_{10}(5) + \log_{10}(4)$

e) $\dfrac{1}{2} \cdot \log_4(16) - \dfrac{1}{3}\log_4(8)$

f) $2 \cdot \log_8\left(\sqrt[4]{8}\right)$

7.7 Exponential and logarithmic equations

Let's see two different cases:

Example.

$$\begin{array}{rcl|l} \log_6(4x-5) & = & \log_6(2x+1) & \\ 4x-5 & = & 2x+1 & -2x+5 \\ 2x & = & 6 & :2 \\ x & = & 3. & \end{array}$$

It is important to check that replacing x by 3 in the initial equation does not lead to the calculation of the logarithm of 0 or a negative number. In this case, the logarithm of 7 is obtained on both sides of the initial equation.

Hence, $x = 3$ is a valid solution.

Example.

$$\begin{array}{rcl|l} \log_4(5+x) & = & 3 & \text{Exponential form} \\ 5+x & = & 4^3 & \\ 5+x & = & 64 & -5 \\ x & = & 59. & \end{array}$$

Once checked, $x = 59$ is indeed the solution of the equation.

Exercise 7.10. Solve.

a) $\log_4(x) = \log_4(8-x)$

b) $\log_5(x-2) = \log_5(3x+7)$

c) $\log_{10}(x-12) = \log_{10}(-3x-4)$

d) $\log_4(x+4) = \log_4(1-x)$

e) $\log_{10}(2x-3) = \log_{10}(3x+1)$

f) $\log_3(x-4) = 2$

g) $\log_2(x-5) = 4$

h) $\log_9(x) = \dfrac{1}{2}$

i) $\log_4(3x+3) = \log_4(5) + \log_4(3)$

j) $\ln(2x-6) = \ln(12) - \ln(3)$

k) $3\log_{10}(x) = 2\log_{10}(8)$

l) $\log_2(x^2-4) = 2 \cdot \log_2(x+3)$

Exercise 7.11. Solve the following equations.

a) $17^x = 4'913$

b) $10^x = 350$

c) $100 \cdot 9^x = 6'561$

d) $3 \cdot e^x - 250 = 1'625$

7.8 Change of base

Calculating $\log_3(9)$ does not require a calculator, you can easily find the result. However, it gets more complicated if you are looking for a less obvious logarithm like $\log_5(8)$. Although the calculator has a LOG key, it is defined with base 10. The LN key is, obviously, defined with base e. To successfully calculate an approximation of $\log_5(8)$, we must change the base.

Theorem (Base).

$$\boxed{\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}}$$

where b and x are strictly positive numbers; b different from 1.

Proof. Let's define $m = \log_b(x)$.
Using the exponential form, we have

$$b^m = x.$$

If we apply logarithm with base 10 on both side of the equal sign, we get

$$\begin{array}{rcl|l} \log(b^m) & = & \log(x) & \text{Property 4} \\ m \cdot \log(b) & = & \log(x) & : \log(b) \\ m & = & \dfrac{\log(x)}{\log(b)}. & \end{array}$$

□

Exercise 7.12. Compute the following logarithms:

a) $\log_7(200)$ b) $\log_{5,1}(34,7)$

c) $\log_{25}(125)$ d) $\log_{49}(2'401)$

7.9 Applications

The logarithm is very useful when you want to solve an equation where the unknown is the exponent.

Example. Let's take the previous example again.
After how many years will 1'000 francs invested in a 2% account reach the sum of 1'268,24 francs?

$$\begin{array}{rcl|l} 1'268,24 & = & 1'000 \cdot (1+0,02)^t & : 1'000 \\ 1,26824 & = & 1,02^t & \text{we change into the logarithmic form} \\ \log_{1,02}(1,26824) & = & t & \text{Change of base formula} \\ t & = & \dfrac{\log(1,26824)}{\log(1,02)} & \\ t & = & 12 \text{ years.} & \end{array}$$

It's also possible to solve like this:

$$\begin{array}{rcl|l} 1'268,24 & = & 1'000 \cdot (1+0,02)^t & : 1'000 \\ 1,26824 & = & 1,02^t & \text{We apply logarithm on both side} \\ \log(1,26824) & = & \log(1,02^t) & \text{Property 4} \\ \log(1,26824) & = & t \cdot \log(1,02) & : \log(1,02) \\ t & = & \dfrac{\log(1,26824)}{\log(1,02)} & \\ t & = & 12 \text{ years.} & \end{array}$$

Exercise 7.13. The island of Manhattan was sold for 24 francs in 1626. How much would this amount have been in 1996 if it had been invested at 6% per year?

Exercise 7.14. How long does it take for a principal of 3'000 francs to reach the amount of 5'000 francs with an interest rate of 5%?

Exercise 7.15. 1 franc was invested in the bank in 1900 at an annual rate of 4%.

a) What amount will there be a century later, i. e. in the year 2000?

b) After how many years will the amount have reached 1'000 francs?

Exercise 7.16. A principal of 2'500 francs is invested in the bank in 2014. Knowing that the interest rate is $2,25\%$, in what year will the capital exceed 7'500 francs?

Exercise 7.17. A principal P_0 is invested in a bank at an annual rate of 5%.

a) What will be the amount after t years?

b) After how many years will it have doubled? Tripled?

Exercise 7.18. 40'000 francs are placed in 2003 in an bank account on January 1st, at an annual interest rate of 5%. In January 2009, a certain amount is withdrawn from this amount to finance a project. Then the entire account is withdrawn in January 2013, and the total amount is 34'768,15 francs. What was the amount withdrawn in 2009?

Exercise 7.19. After how many years, a 125'000 francs principal invested at a rate of 4% overpasses a 250'000 francs principal invested at a rate of 2%?

Exercise 7.20. Between 1990 and 2000, the population of the city A increased from 3'000 to 5'000 inhabitants. During the same period, the population of the city B went from 6'000 to 4'000 inhabitants. It is assumed that the demographic evolution of these two cities is exponential.

a) Determine, in %, the annual rates of variation of these two populations.

b) In what year did the population of city A overtake the population of city B?

Exercise 7.21. As shown in the table below, for the same model policyholder, the health insurance premiums for the two insurance companies Assurmed and Visante have evolved exponentially between 2000 and 2011.

	2000	2011
Assurmed	172,00	264,80
Visante	131,00	236,05

The monthly premiums are in francs.

a) What were the respective annual premium growth rates for this model insured over this period?

b) What were the respective premiums in 2008?

c) From which year will the Assurmed insurance be the most advantageous?

Exercise 7.22 (U.S. population growth). The population $N(t)$ (in millions) of the USA, t years after 1980 can be represented by the formula :

$$N(t) = 227 \cdot 10^{0,0009t}.$$

In what year will the population be twice the population of 1980?

Exercise 7.23. A medicine is eliminated from the body through urine. It is known that, for a 10 milligram dose, the amount $q(n)$ remaining in the body n hours after taking it is given by the relationship

$$q(n) = 10 \cdot 0,8^n.$$

.

a) What is the remaining amount of this medicine after three hours?

b) After how long does 2 mg of this medicine remain in the body?

Exercise 7.24 (Human memory). Mathematics students have taken a test on a subject and retake it every month with a similar test. The average score $f(t)$ for the class is given by the following formula:

$$f(t) = 80 - 17\log_{10}(t+1), 0 \leq t \leq 12$$

where t is the number of month.

a) What was the average score of the very first test?

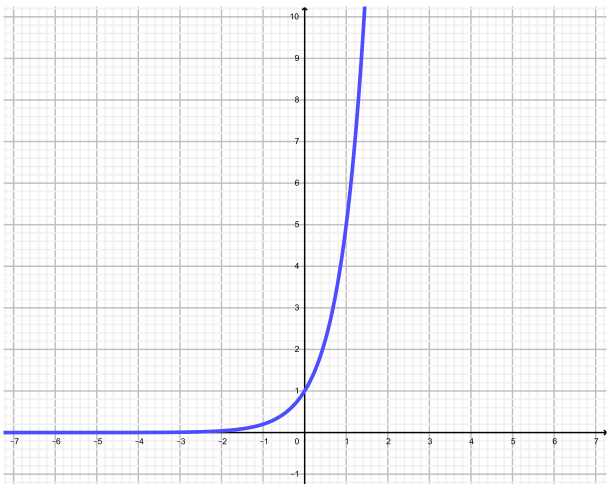
b) What is the average score after 6 months?

c) Determine the number of months elapsed if the average score is $61,5$.

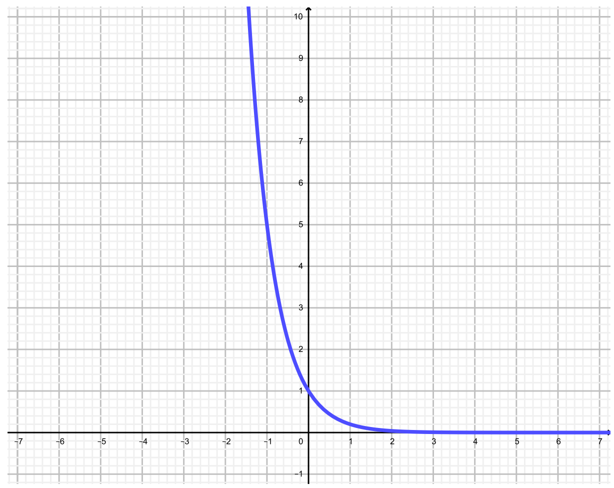
7.10 Solutions

Exercise 7.1.

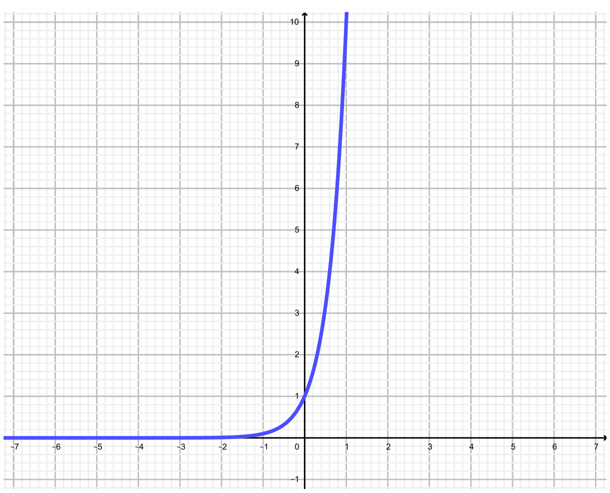
a)



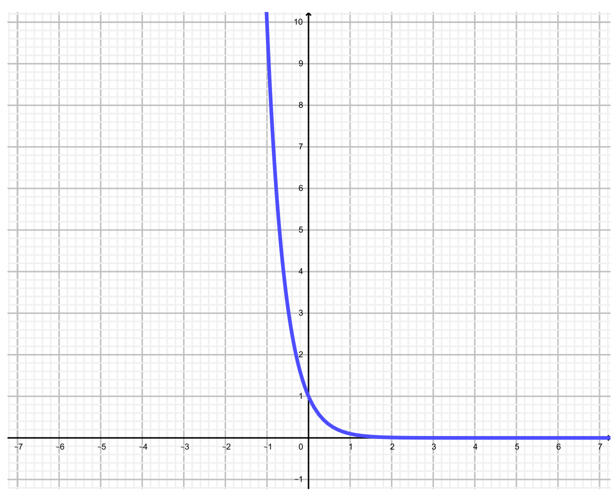
b)



c)



d)



f is increasing if $a > 1$ and decreasing if $0 < a < 1$.

Exercise 7.2.

a) $x = 5$	b) $x = 2$
c) $x = 3$ and $x = -1$	d) $x = 3$
e) $x = 5$	f) $x = 4$
g) $x = 2$ and $x = -\dfrac{1}{2}$	h) $x = 3$
i) $x = \dfrac{18}{5}$	j) $x = -2$
k) $x = 7$	l) $x = -\dfrac{4}{99}$

Exercise 7.3.

a) $\sim 1'340,10$ francs.

b) $\sim 822,70$ francs.

c) $6,5\%$.

Exercise 7.4. $E(5) \cong 231, E(8) \cong 180, E(12) \cong 129$.

Exercise 7.5.

a) $S(t) = 1'000 \cdot 1,14^t$.

b) Approximately 1'482 infected people.

c) Approximately 769 infected people.

Exercise 7.6.

a) $\sim 3,64\%$.

b) $\sim 60'650,54$ m^3.

c) $\sim 93'123,29$ m^3.

Exercise 7.7.

a) 3	b) 3
c) 3	d) 6
e) 10	f) 7
g) 1	h) 0
i) -1	j) -2
k) -3	l) $\frac{1}{2}$
m) $\frac{1}{2}$	n) $\frac{1}{9}$
o) 3	p) -2

Exercise 7.8.

a) 16	b) 243
c) 64	d) 4
e) 5	f) 10

Exercise 7.9.

a) 2	b) -3
c) 3	d) 2
e) $\frac{1}{2}$	f) $\frac{1}{2}$

Exercise 7.10.

a) $x = 4$	b) No solution
c) No solution	d) $x = -\frac{3}{2}$
e) No solution	f) $x = 13$
g) $x = 21$	h) $x = 3$
i) $x = 4$	j) $x = 5$
k) $x = 4$	l) $x = -\frac{13}{6}$

Exercise 7.11.

a) $x = 3$	b) $x \cong 2,544$
c) $x \cong 1,904$	d) $x \cong 6,438$

Exercise 7.12.

a) $\sim 2,723$	b) $\sim 2,177$
c) $\frac{3}{2}$	d) 2

Exercise 7.13. 55’383’626’485,90 francs.

Exercise 7.14. 11 years.

Exercise 7.15.

a) 50,50 francs.

b) After 177 years.

Exercise 7.16. The amount overpasses 7’500 francs in 2064.

Exercise 7.17.

a) $P(t) = P_0 \cdot 1,05^t$.

b) It will have doubled after 15 years and tripled after 23 years.

Exercise 7.18. 25’000 francs.

Exercise 7.19. After 36 years.

Exercise 7.20.

a) The city A has a growth rate of $5,24\%$ and the city B has a decline rate of 4%.

b) In 1998.

Exercise 7.21.

a) 4% for Assurmed and $5,5\%$ for Visante.

b) $235,4$ francs for Assurmed and $201,05$ francs for Visante.

c) From 2020.

Exercise 7.22. In 2315.

Exercise 7.23.

a) $5,12$ mg.
b) After 8 hours.

Exercise 7.24.

a) 80 points.
b) 65,63 points.
c) 11,25 months.

7.11 Chapter objectives

At the end of this chapter, the student should be able to

7.1 ☐ Sketch the graph of an exponential function.
7.2 ☐ Solve a simple exponential equation.
7.3 ☐ Compute simple logarithms.
7.4 ☐ Sketch the graph of a logarithmic function.
7.5 ☐ Solve a simple logarithmic equation.
7.6 ☐ Solve any exponential equation.
7.7 ☐ Compute a logarithm using the change of base formula.
7.8 ☐ Solve a problem using the compound interest formula.
7.9 ☐ Solve other problems involving exponential functions and logarithmic functions.

Chapter 8

Inequations

8.1 Introduction

Definition. An *inequation* is an inequality ($<$, $\leq$, $>$ or $\geq$) between two algebraic expressions.

Example. One student received the grades of 3, 4 and 3.5 on his first three written assignments. What grade should the student obtain in the fourth (and final) written assignment to ensure an sufficient average?

Resolution. Let x be the grade of the last in the final assignment.
The average will be given by

$$\frac{3+4+3,5+x}{4} = \frac{10,5+x}{4}.$$

For a student to get a sufficient GPA (Grade Point Average), the average must be $3,75$ or higher. This condition results in a *first degree inequation with one unknown*

$$\frac{10,5+x}{4} \geq 3,75.$$

A *solution* of this inequation is a number x such that the inequality is satisfied. Hence, $x = 6$ is a solution of this inequation, because

$$\frac{10,5+6}{4} = \frac{16,5}{4} = 4,125 \geq 3,75.$$

It's the same for $x = 5,5$, because

$$\frac{10,5+5,5}{4} = \frac{16}{4} = 4 \geq 3,75.$$

On the other hand, $x = 1$ is not a solution of this inequation, because

$$\frac{10,5+1}{4} = \frac{11,5}{4} = 2,875 < 3,75.$$

Solving an inequation means finding all the numbers that verify the inequality:

$$\begin{array}{rcl|l} \frac{10,5+x}{4} & \geq & 3,75 & \cdot 4 \\ 10,5+x & \geq & 15 & -10,5 \\ x & \geq & 4,5. & \end{array}$$

Therefore, this student will have to obtain at least $4,5$ in order to finish the semester with a sufficient average. The set S_i of solutions can also be written as follows

$$S_i = [4,5; +\infty[.$$

Remark. $[4,5;+\infty[$ is the set of solutions of the inequation. Since the highest score that can be obtained is 6, all solutions strictly above 6 should be rejected. Thus, the set S_p of the solutions of the problem (but not of the inequation!) is given by

$$S_p = [4,5;6].$$

8.2 First degree inequation with one unknown

An inequation is like an equation where we replace the "=" sign with an *inequality sign* . There are four of them:

— $>$: *greater than*

— $<$: *less than*

— $\geq$: *greater than or equal to*

— $\leq$: *less than or equal to*

Solving an inequation is similar to solving an equation. With one exception:

$$\boxed{\text{When multiplying or dividing by a negative number, we reverse the inequation sign.}}$$

Indeed,

$$\begin{array}{rcl|l} 3 & \geq & 2 & \cdot(-2) \\ -6 & \leq & -4. & \end{array}$$

Example.

1.
$$\begin{array}{rcl|l} 5(x-2)-4(2x-3) & \geq & 40 & \\ 5x-10-8x+12 & \geq & 40 & \\ -3x+2 & \geq & 40 & -2 \\ -3x & \geq & 38 & :(-3) \\ x & \leq & -\dfrac{38}{3}. & \end{array}$$

2.
$$\begin{array}{rclcl|l} -6 & < & 2x-3 & \leq & 5 & +3 \\ -3 & < & 2x & \leq & 8 & :2 \\ -\dfrac{3}{2} & < & x & \leq & 4. & \end{array}$$

Exercise 8.1. True or false? Justify your answer.

a) -4 is a solution of the inequation $-3x > 7$.

b) 8 is a solution of the inequation $5x - 3 \leq 37$.

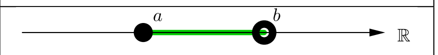
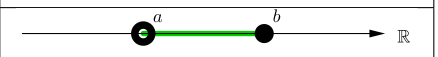
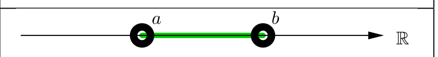
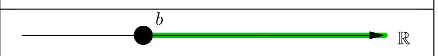
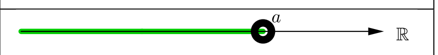
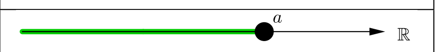
c) 2 is a solution of the inequation $3x - 2 < -4x + 12$.

d) 5 is a solution of the inequation $5x - 3 \geq 5x + 2$.

8.3 Intervals

Intervals are simple notations to describe certain subsets of $\mathbb{R}$. In particular, they are used to give sets of solutions of inequations.

In the table below, where a and b are two real numbers where $a < b$, each line describes the same subset of $\mathbb{R}$ in three equivalent ways.

Inequality	Interval	Graphical representation
$a \leq x \leq b$	$[a; b]$	
$a \leq x < b$	$[a; b[$	
$a < x \leq b$	$]a; b]$	
$a < x < b$	$]a; b[$	
$x > b$	$]b; +\infty[$	
$x \geq b$	$[b; +\infty[$	
$x < a$	$]-\infty; a[$	
$x \leq a$	$]-\infty; a]$	

Remark. From now on, all the solutions of an inequation will be expressed in the form of an interval.

Example. All the solutions of the inequations in the previous example are written as follows

1. $S = \left]-\infty; -\frac{38}{3}\right]$.
2. $S = \left]-\frac{3}{2}; 4\right]$.

Exercise 8.2. Write the following sets as intervals.

a) All the numbers greater or equal to -3 and less or equal to 5.

b) All the numbers greater or equal to 4 and less than 5.

c) All the numbers less than 1.

d) All the numbers greater or equal to 10.

e) All the numbers greater than -2 and less than 2.

f) All the strictly positive numbers.

g) All the numbers less or equal to 0.

h) All the real numbers.

Exercise 8.3. Solve the following inequations and give the set of solutions as an interval.

a) $x - 7 > -3x + 1$

b) $3 - 2x > 3x - 5$

c) $5(1 + 4x) > 7 + 12x$

d) $7(x - 3) > 7x + 5$

e) $4x - 2 < 4(x + 10)$

f) $-2x + \dfrac{x}{2} + 1 \leq -\dfrac{x}{4}$

g) $4(x^2 + 1) + 8(3x - 4) > 4x^2$

h) $(3x + 45)(3x + 3) < (3x + 6)(3x + 18)$

i) $\dfrac{3 - 4x}{2} < \dfrac{x - 3}{4}$

j) $\dfrac{2x - 3}{3} - \dfrac{x + 4}{5} \leq \dfrac{x + 2}{4}$

Exercise 8.4. Solve the following double inequalities and give the set of solutions as an interval.

a) $-3 < 2x - 5 < 7$

b) $-2 < 3 + \frac{1}{4}x \leq 5$

c) $0 \leq 4 - \frac{1}{3}x \leq 2$

d) $3 \leq \frac{2-3x}{5} \leq 7$

Exercise 8.5. Solve the following system of inequations and give the set of solutions as an interval.

a) $\left\{ \begin{array}{rcl} 5x - 10 & < & 4x \\ 5 + 2x & \geq & 7 \end{array} \right.$

b) $\left\{ \begin{array}{rcl} 1 - x & \geq & x - 5 \\ x(2-x) & < & 4 - x^2 \end{array} \right.$

Exercise 8.6. The following formula connects the Fahrenheit and Celsius temperature.

$$C = \frac{5}{9}(F - 32).$$

In Switzerland, on January 26, 2005, the temperature was between $-30.4°\mathrm{C}$ and $-3°\mathrm{C}$. Express the situation in Fahrenheit.

8.4 Linear inequalities with two unknowns

These are inequalities of the form $ax + by > c$, $ax + by < c$, $ax + by \geq c$ or $ax + by \leq c$. It is more tricky to express the solutions using intervals because the values that x can take depend directly on y (and vice-versa). In order to best illustrate the set of solutions of such an inequality, we colour the set of points of the (Euclidean) plane satisfying the linear inequality.

Example. Let's solve the inequality

$$3x + 2y < 6.$$

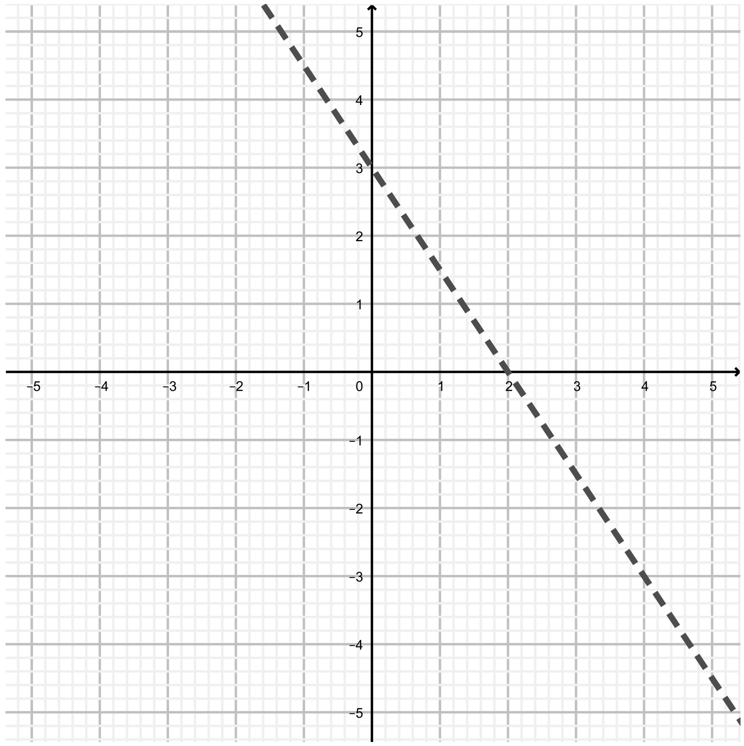
The first thing to do is to isolate the variable y in order to be able to draw the border of the colored area in an easier way.

$$\begin{array}{rcl|l} 3x + 2y & < & 6 & -3x \\ 2y & < & -3x + 6 & :2 \\ y & < & -\frac{3}{2}x + 3. & \end{array}$$

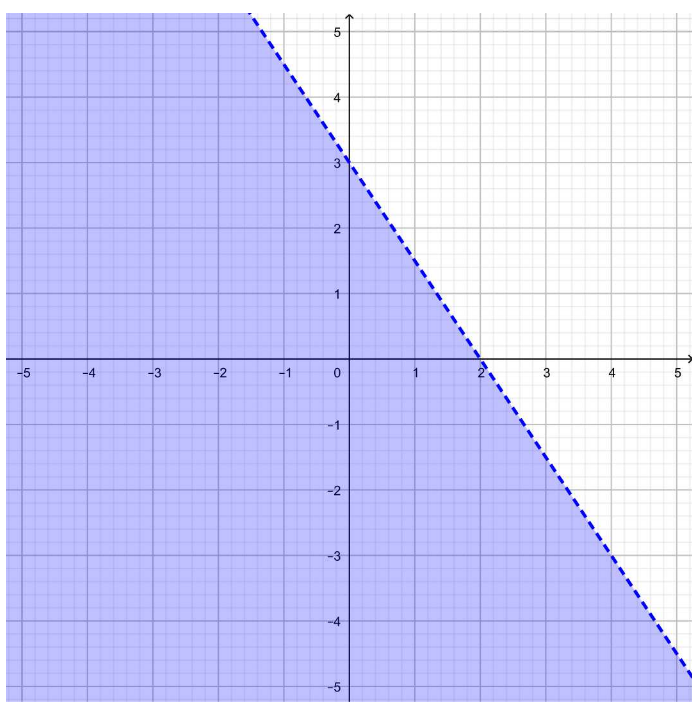
Let's take this inequality and draw the line that we would get if there was a "=" instead of the "<".

Convention.

- If the inequality sign is "strictly greater than" or "strictly less than", i.e. $>$ or $<$, then we draw a dashed line.
- If the inequality sign is "greater than or equal to" or "less than or equal to", i.e. $\geq$ or $\leq$, then the line is drawn as a full line.



This right here is called *border*. In fact, it defines the limit between the zone containing the solutions and the rest. How do we know which is the correct zone? Just use the inequation $y < -\dfrac{3}{2}x + 3$. We observe that y must be less than $-\dfrac{3}{2}x + 3$. As the value of y must be less than the rest, then we color the area below the line (we would have colored above if the sign was $>$ or $\geq$).



All the points that can be found in the colored area will satisfy the inequation

$$3x + 2y < 6.$$

The dashed line illustrates the fact that the points on the line are not included in the solutions. However, if the line were full, the points on the line would be included.

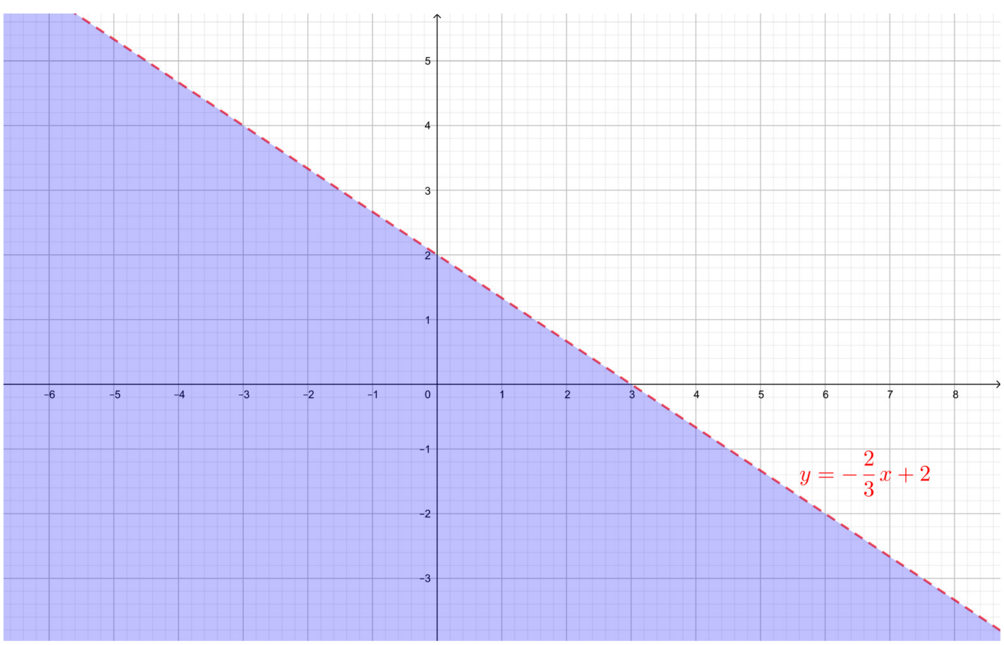
Example. Let's solve this system of inequalities

$$\begin{cases} 2x + 3y & < & 6 \\ 4x + 2y & \geq & -12 \\ x & \geq & -4 \\ y & \geq & -2 \end{cases}.$$

To represent the set of solutions of this system, it is first necessary to isolate the variable y from the inequation $2x + 3y < 6$:

$$\begin{array}{rcl|l} 2x + 3y & < & 6 & -2x \\ 3y & < & -2x + 6 & :3 \\ y & < & -\dfrac{2}{3}x + 2. & \end{array}$$

The set of solutions of the inequation $2x + 3y < 6$ is the set of points below the line $y = -\dfrac{2}{3}x + 2$.



To represent all the solutions of the second inequation, it is possible to proceed in a similar way. However, we will use a different approach, by calculating the coordinates of two points of the line $4x + 2y = -12$.

— For the first point, we put $x = 0$:

We have

$$\begin{aligned} 4 \cdot 0 + 2y &= -12 \\ 2y &= -12 \\ y &= -6. \end{aligned}$$

So the point is $A(0; -6)$.

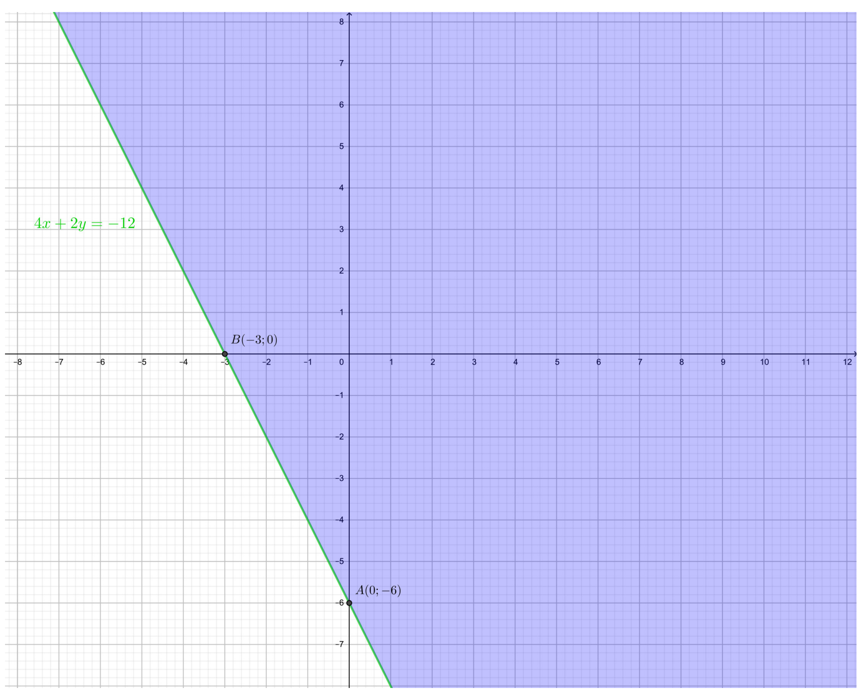
— For the second point, we put $y = 0$:

We have

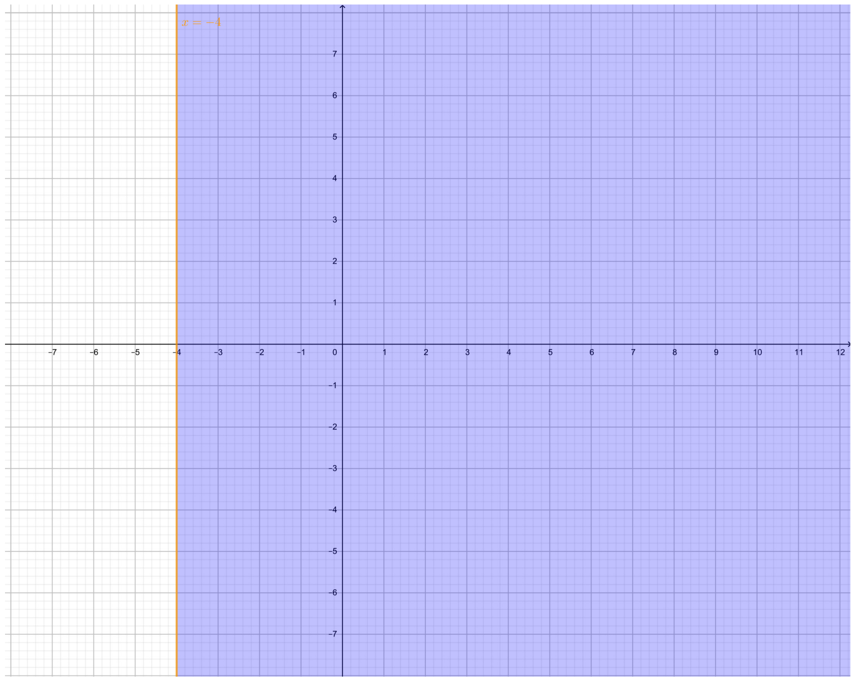
$$\begin{aligned} 4x + 2 \cdot 0 &= -12 \\ 4x &= -12 \\ x &= -3 \end{aligned}$$

So the point is $B(-3; 0)$.

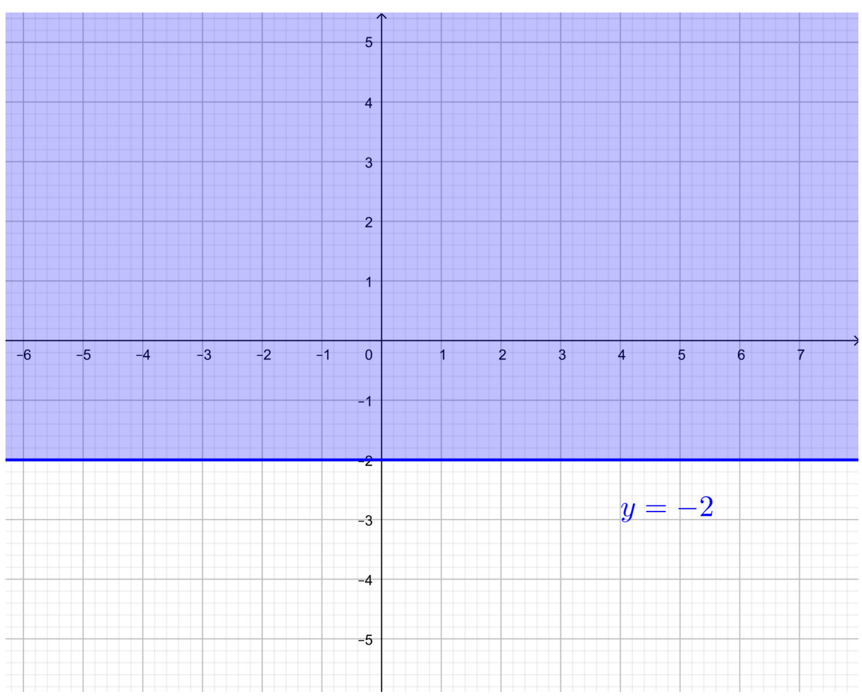
Hence, the line $4x + 2y = -12$ passes through the points $A(0; -6)$ and $B(-3; 0)$.
The solutions of the inequation $4x + 2y \geq -12$ are in the area above the line $4x+2y = -12$.



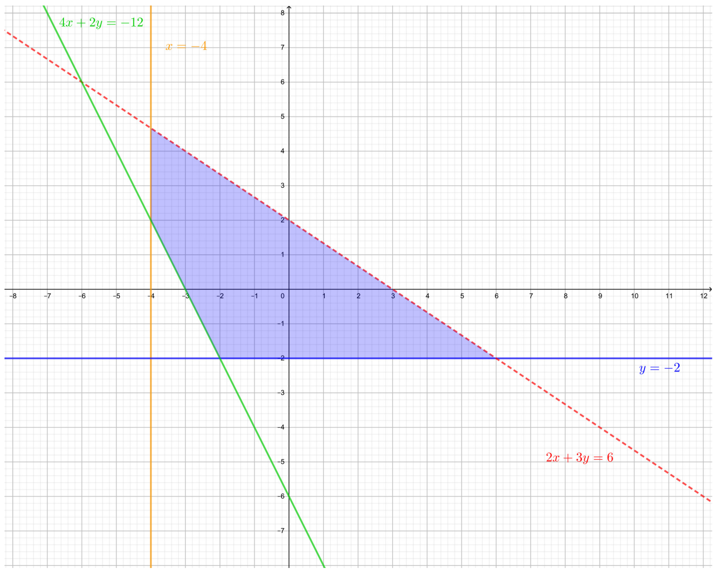
The set of solutions of the inequation $x \geq -4$ is the area on the right of the line $x = -4$.



Finally, the set of solutions of the inequation $y \geq -2$ is the area above the line $y = -2$.



Overall, the system's solution set is the area that is the solution of the four inequalities:



Exercise 8.7. Solve graphically.

a) $x+3y \geq 15$ b) $2x+5y < 20$
c) $3x+7y > 63$ d) $6x+5y \leq 120$

Exercise 8.8. Solve graphically.

a) $\begin{cases} y \leq x+3 \\ y \geq -\frac{1}{4}x+2 \end{cases}$ b) $\begin{cases} x+3y \geq 0 \\ 2x+y-5 < 0 \end{cases}$

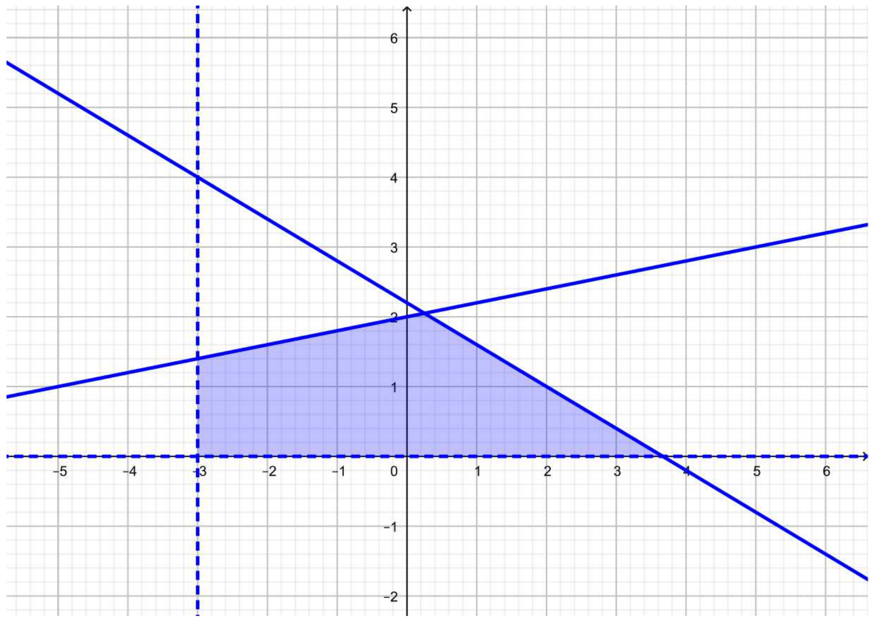
c) $\begin{cases} x \geq 8-2y \\ x > 2-2y \end{cases}$ d) $\begin{cases} y \leq 2x+10 \\ y \geq x+1 \\ y > -2x-2 \end{cases}$

e) $\begin{cases} y \geq 2x-4 \\ y \geq \frac{2}{3}x+4 \\ y \geq -4x+4 \end{cases}$ f) $\begin{cases} y > \frac{x}{4}+2 \\ y \geq x-1 \\ y > -\frac{3}{4}x+6 \end{cases}$

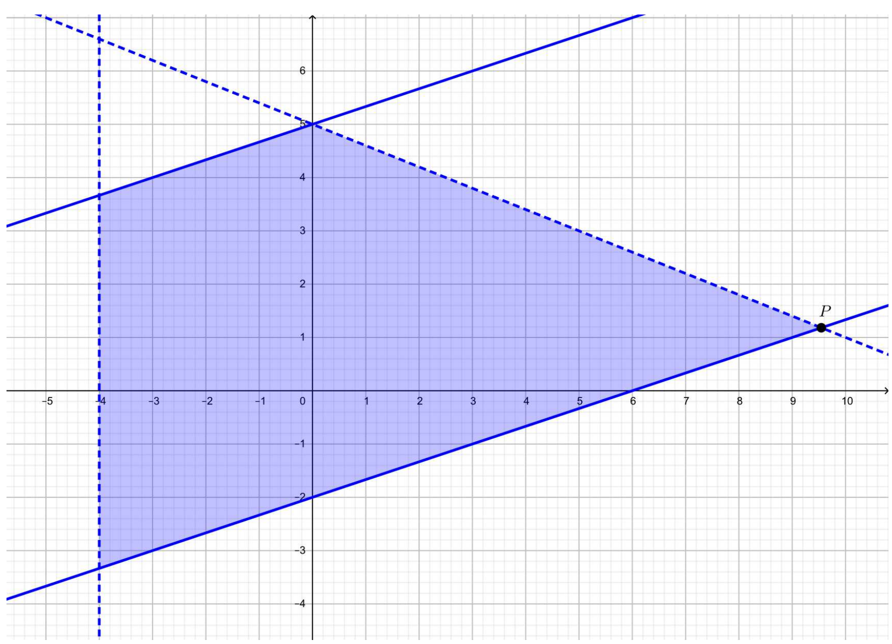
g) $\begin{cases} 5x+9y < 90 \\ 4x+y < 48 \\ x \geq 0 \\ y \geq 0 \end{cases}$ h) $\begin{cases} 5x+7y > 35 \\ 2x+y > 10 \\ x \geq 0 \\ y \geq 0 \end{cases}$

i) $\begin{cases} 12x+5y \leq 60 \\ 4x+3y \leq 24 \\ x \geq 0 \\ y \geq 0 \end{cases}$ j) $\begin{cases} 2x+5y \geq 30 \\ x+y \geq 10 \\ 3x+2y \geq 24 \\ x \geq 0 \\ y \geq 0 \end{cases}$

Exercise 8.9. Determine a system of inequalities whose solution set is the colored area below.



Exercise 8.10. Determine a system of inequalities whose solution set is the colored area below.



Compute the exact coordinates of point P and determine if point P belongs to the solution set (justify your answer).

8.5 Solutions

Exercise 8.1.

a) True b) True
c) False d) False

Exercise 8.2.

a) $[-3;5]$ b) $[4;5[$
c) $]-\infty;1[$ d) $[10;+\infty[$
e) $]-2;2[$ f) $]0;+\infty[$
g) $]-\infty;0]$ h) $]-\infty;+\infty[$

Exercise 8.3.

a) $x \in]2;+\infty[$ b) $x \in \left]-\infty;\frac{8}{5}\right[$
c) $x \in \left]\frac{1}{4};+\infty\right[$ d) No solution
e) $x \in \mathbb{R}$ f) $x \in \left[\frac{4}{5};+\infty\right[$
g) $x \in \left]\frac{7}{6};+\infty\right[$ h) $x \in \left]-\infty;-\frac{3}{8}\right[$
i) $x \in]1;\infty[$ j) $x \in \left]-\infty;\frac{138}{13}\right]$

Exercise 8.4.

a) $x \in]1;6[$ b) $x \in]-20;8]$
c) $x \in [6;12]$ d) $x \in \left[-11;-\frac{13}{3}\right]$

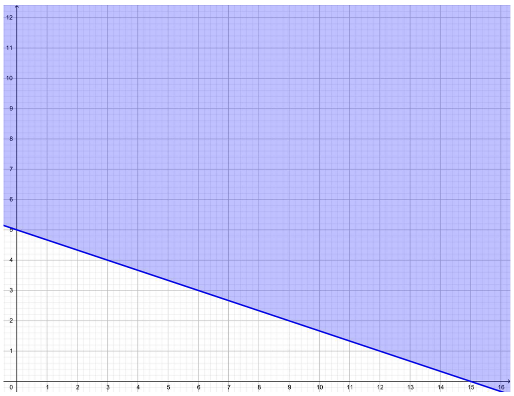
Exercise 8.5.

a) $x \in [1;10[$ b) $x \in]-\infty;2[$

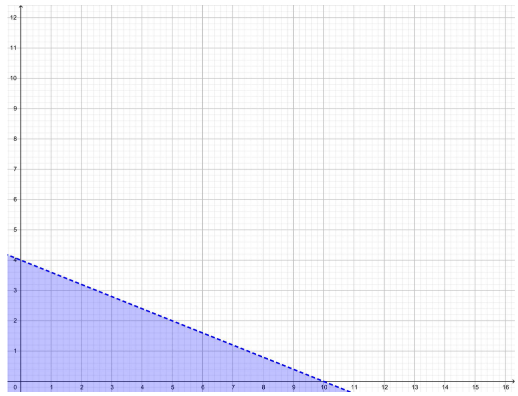
Exercise 8.6. $-22,72 \leq F \leq 26,6$.

Exercise 8.7.

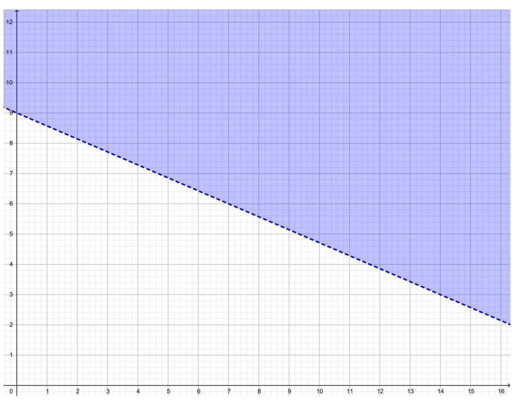
a)



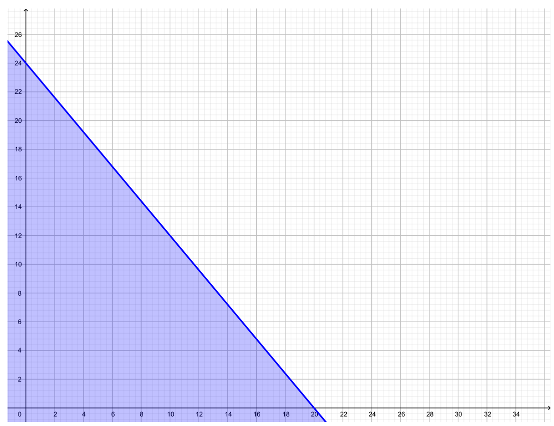
b)



c)

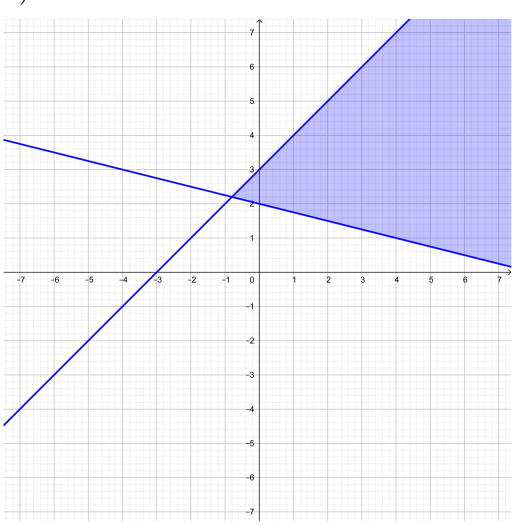


d)

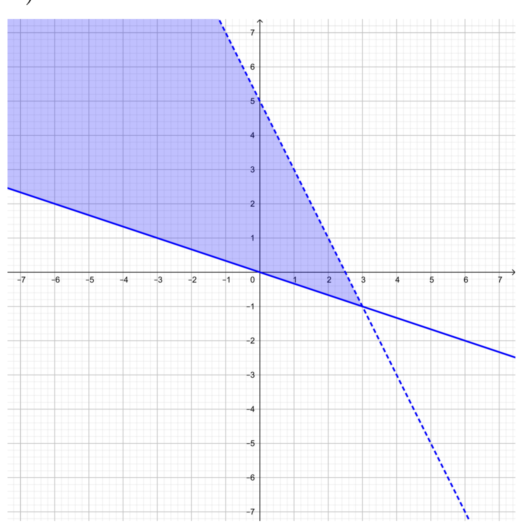


Exercise 8.8.

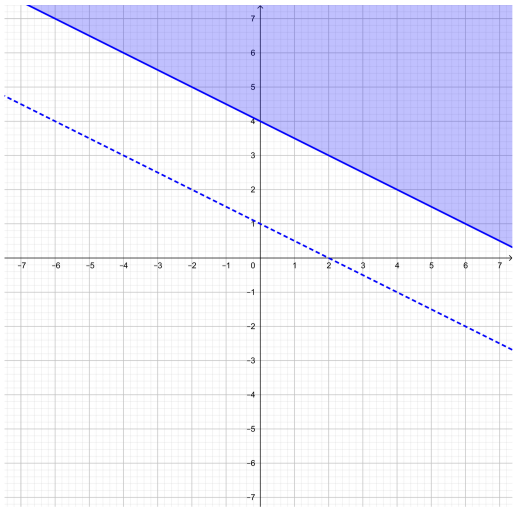
a)



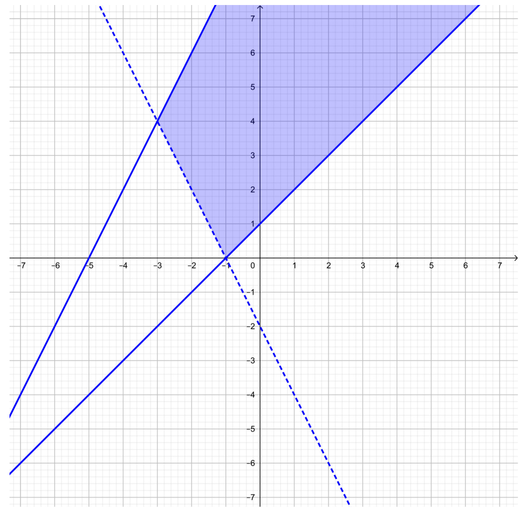
b)



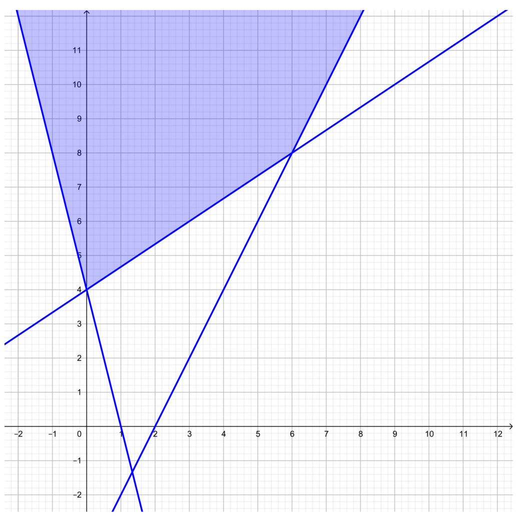
c)



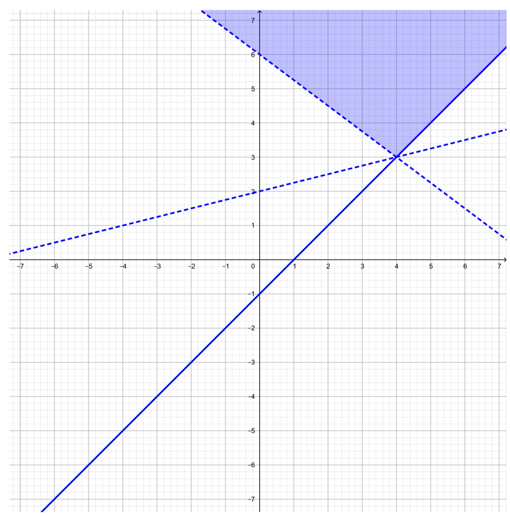
d)



e)



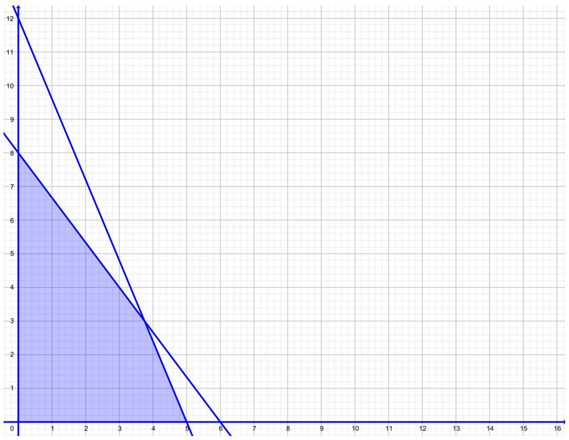
f)



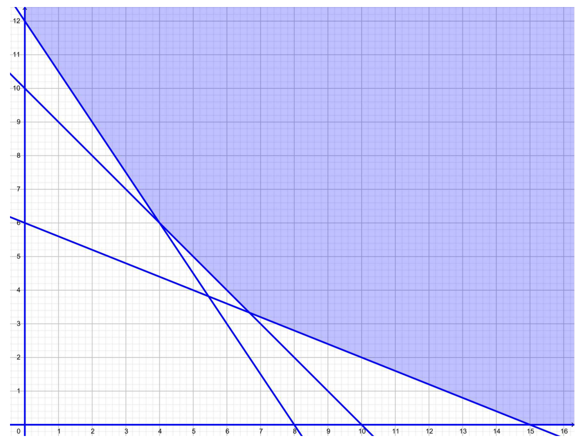
g)

h)

i)



j)



Exercise 8.9.

$$\begin{cases} x & > & -3 \\ y & \leq & -\dfrac{3}{5}x + \dfrac{11}{5} \\ y & \leq & \dfrac{1}{5}x + 2 \\ y & > & 0 \end{cases}.$$

Exercise 8.10.

$$\begin{cases} x & > & -4 \\ y & \leq & \dfrac{1}{3}x + 5 \\ y & < & -\dfrac{2}{5}x + 5 \\ y & \geq & \dfrac{1}{3}x - 2 \end{cases}.$$

$P(9,\overline{54};1,\overline{18})$ does not belong to the solution set, because it is on, at least, one dashed line.

8.6 Chapter objectives

At the end of this chapter, the student should be able to

8.1 ☐ Solve a simple first degree inequation and give the answer as an interval.
8.2 ☐ Solve a double inequality and give the answer as an interval.
8.3 ☐ Solve a system of inequalities with one unknown and give the answer as an interval.
8.4 ☐ Graphically solve a linear inequation.
8.5 ☐ Graphically solve a system of inequalities with two unknowns.
8.6 ☐ Determine the system of inequalities from its drawn solution set.

Chapter 9

Linear programming

9.1 Introduction

During the Second World War, the United States Air Force of America had many problems concerning the allocation of its resources (human and material). Naturally, many specialists looked into the problem and one of them was George Danzig. Shortly after the war, in 1946, Danzig developped a more general formulation of this kind of problems and proposed a method of solving them, the *method of the simplex.*

This general problem can be formulated as follows: to find the *maximum* (or *minimum*) value of a function with several variables if these variables are subjected to *constraints.* For example, let's suppose that a company manufactures various products and that for each of these products there are costs of different manufacturing processes in terms of labour and raw materials. The company knows the profit it makes by selling each of these products. The company must then ask itself the following question: how many product of each type must be manufactured to obtain the maximum overall profit? In general, such problems can be quite complex. However, if the function to be optimized, i.e. to render maximum or minimum, is linear and if the constraints can be expressed by inequalities, we can develop a theory simple enough to solve problems like this, we call it *linear programming.* We'll limit ourselves to problems with only two variables. This will allow us to illustrate the solution with a simple graphical representation.

9.2 Linear optimization with two variables

Example. One company manufactures two types of metal boxes. Manufacturing a type A box requires 1 hour of work and 3 kg of metal, while box B requires 2 hours of work and 2 kg of metal. The company can work 80 hours and produce 120 kg of metal. Knowing that, for a box, the profit is 20 francs for the type A and 30 francs for the type B, how should the production be organized in order to maximize the profit?

This statement can be summarized using the table below.

	Work	Metal	Profit
Type A	1 h	3 kg	20 Frs
Type B	2 h	2 kg	30 Frs
Total	≤ 80 h	≤ 120 kg	P

If x is the number of boxes of type A and y is the number of boxes of type B, the problem means finding the maximum value of the expression

$$P = 20x + 30y$$

with the constraints

$$\begin{cases} x + 2y & \leq & 80 \\ 3x + 2y & \leq & 120 \\ x & \geq & 0 \\ y & \geq & 0 \end{cases}.$$

Let us then represent the lines $x + 2y = 80$, $3x + 2x = 120$, $x = 0$ and $y = 0$. To do this, two points per line should be calculated.

1. Points of $x + 2y = 80$:

   — For the first point, we put $x = 0$:
   We have
   $$\begin{array}{rcl} 0 + 2y & = & 80 \\ 2y & = & 80 \\ y & = & 40. \end{array}$$
   So, $(0; 40)$.

   — For the second point, we put $y = 0$:
   We have
   $$\begin{array}{rcl} x + 2 \cdot 0 & = & 80 \\ x & = & 80. \end{array}$$
   So, the point is $(80; 0)$.

   Hence, the line $x + 2y = 80$ passes through $(0; 40)$ and $(80; 0)$.

2. Points of $3x + 2y = 120$:

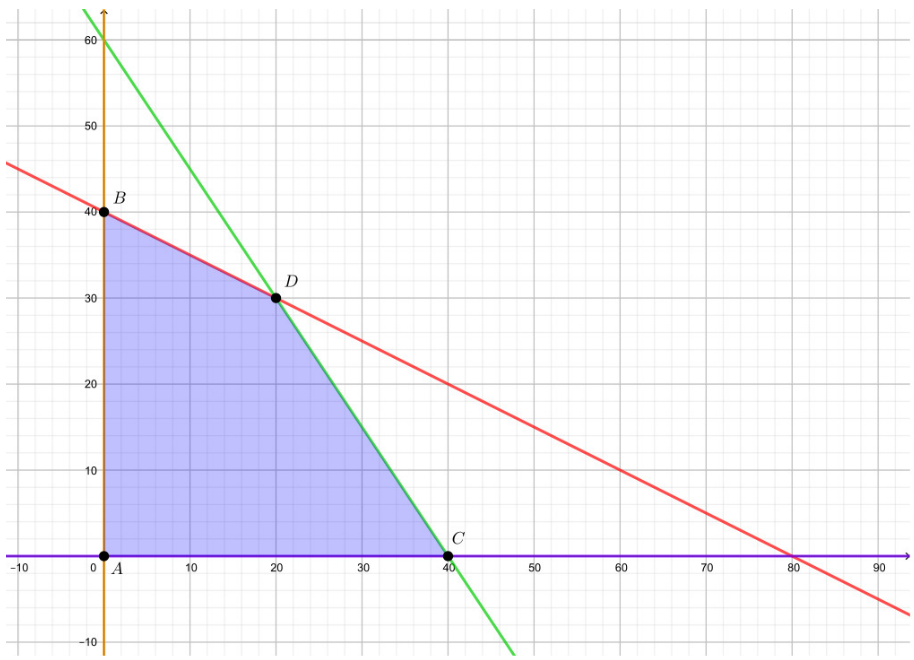
   — For the first point, we put $x = 0$:
   We have
   $$\begin{array}{rcl} 3 \cdot 0 + 2y & = & 120 \\ 2y & = & 120 \\ y & = & 60. \end{array}$$
   So, the point is $(0; 60)$.

   — For the second point, we put $y = 0$:
   We have
   $$\begin{array}{rcl} 3x + 2 \cdot 0 & = & 120 \\ 3x & = & 120 \\ x & = & 40. \end{array}$$
   So, the point is $(40; 0)$.

Hence, the line $3x + 2y = 120$ passes through $(0; 60)$ and $(40; 0)$.

3. Points of $x = 0$ and of $y = 0$:

   The line $x = 0$ is a vertical line passing through the origin. The line $y = 0$ is a horizontal line passing through the origin.



The domain defined by the constraints is the colored quadrilateral above.

It can then be seen that the maximum value sought is necessarily obtained on a vertex of the quadrilateral.

Let us then determine the coordinates of these vertices:

It is clear that the points $A(0; 0)$, $B(0; 40)$ and $C(40; 0)$ are vertices of the quadrilateral.

Now we just need to determine the coordinates of the last vertex D.

To do this, we solve

$$\begin{array}{rrcl} & 3x + 2y & = & 120 \\ - & x + 2y & = & 80 \\ \hline & 2x & = & 40 \\ & x & = & 20. \end{array}$$

We get y when solving

$$\begin{array}{rcl|l} 20+2y & = & 80 & -20 \\ 2y & = & 60 & :2 \\ y & = & 30. & \end{array}$$

So, the vertex is $D(20;30)$.

To find the maximum value of P, we will calculate its value for each vertex of the quadrilateral. We obtain the following values:

Vertex $(x;y)$	Value of $P=20x+30y$
$A(0;0)$	$20\cdot 0+30\cdot 0=0$
$B(0;40)$	$20\cdot 0+30\cdot 40=1'200$
$C(40;0)$	$20\cdot 40+30\cdot 0=800$
$D(20;30)$	$20\cdot 20+30\cdot 30=1'300$

So, the maximum value $P=1'300$ is obtained for $x=20$ and $y=30$. The maximum profit of 1'300 francs is therefore achieved with 20 boxes of type A and 30 of type B.

Exercise 9.1. After a reorganization in the company, the profits are now adjusted to 50 francs for the type A and 20 francs for the type B. Is it then necessary to modify the production plan in order to maximize profit?

Example. A company manufactures both hand decorated plates and vases. The manufacturing time is 2 hours per plate and 3 hours per vase. The decoration time is half an hour for a plate and 2 hours for a vase. The daily production of plates is limited to 30 units. The production workshop has 12 workers who work 8 hours a day. The decoration workshop is composed of 7 workers working 7 hours a day.

Given that the net profit is 70 francs per plate and 160 francs per vase, determine the production ensuring the maximum profit.

The above information can be summarised in the table below.

	Manufacturing	Decoration	Production	Profit
Plates	2 h	$0,5$ h	≤ 30	70 Frs
Vases	3 h	2 h		160 Frs
Total	$\leq 12\cdot 8=96$ h	$\leq 7\cdot 7=49$ h		P

If x is the number of plates and y is the number of vases, the problem consists in finding the maximum value of the expression

$$P=70x+160y.$$

with the constraints

$$\left\{\begin{array}{rcl} 2x+3y & \leq & 96 \\ 0,5x+2y & \leq & 49 \\ x & \leq & 30 \\ x & \geq & 0 \\ y & \geq & 0 \end{array}\right. .$$

In order to determine the domain defined by the constraints, it is necessary to draw the borders given by the equations $2x+3y=96$, $0,5x+2y=49$, $x=30$, $x=0$ and $y=0$. To do this, let's start isolating y in order to identify the slope and the y-intercept of the lines.

1. $2x+3y=96$:

$$\begin{array}{rcl|l} 2x+3y & = & 96 & -2x \\ 3y & = & -2x+96 & :3 \\ y & = & -\frac{2}{3}x+32 & \end{array}$$

   Hence, we can sketch the line using its y-intercept and its slope.

2. $0,5x+2y=49$:

$$\begin{array}{rcl|l} 0,5x+2y & = & 49 & -0,5x \\ 2y & = & -0,5x+49 & :2 \\ y & = & -0,25x+24,5 & \end{array}$$

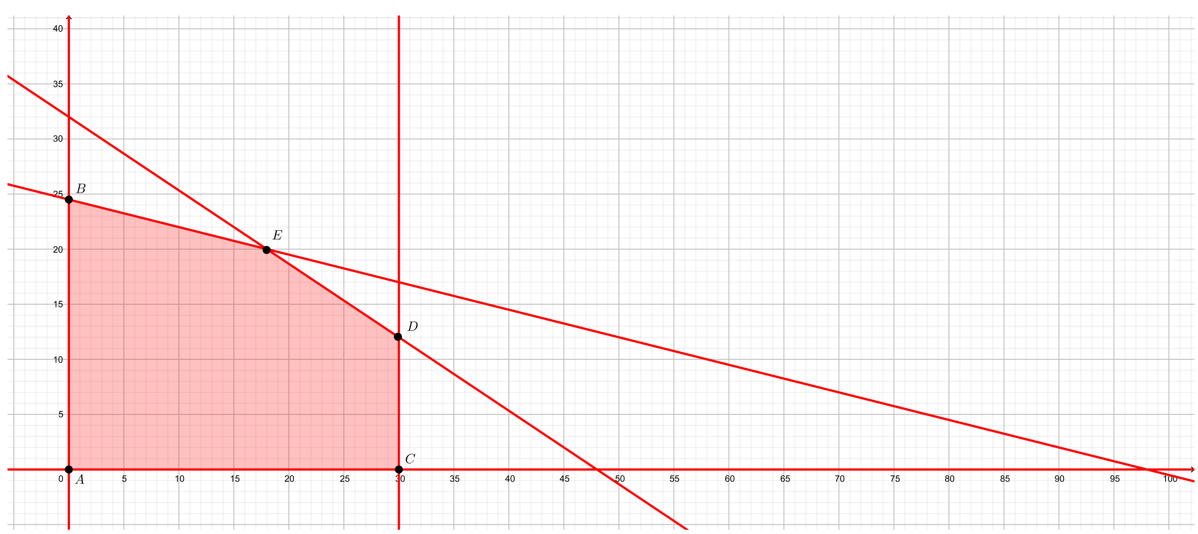
   As before, it is now possible to draw the line.

3. $x=30$:

   It's the vertical line passing through $(30;0)$

4. $x=0$ and $y=0$:

   The line $x=0$ is the vertical line passing through the origin. The line $y=0$ is an horizontal one passing through the origin.



The domain defined by the constraints is shown above.

Same as the previous example. The maximum we're looking for is bound to be reached on a vertex of the polygon.

Let's determine the coordinates of these vertices.

It's clear that points $A(0;0)$, $B(0;24,5)$ and $C(30;0)$ are vertices of the polygon.

Now we just need to determine the coordinates of the last two vertices D and E.
To determine the coordinates of D, we set $x = 30$ and solve

$$\begin{array}{rcl|l} 2\cdot 30 + 3y &=& 96 & \\ 60 + 3y &=& 96 & -60 \\ 3y &=& 36 & :3 \\ y &=& 12. & \end{array}$$

So the vertex is $D(30;12)$.

Finally, let's determine the coordinates of E.
We solve

$$\left\{\begin{array}{rcl} 2x + 3y &=& 96 \\ 0,5x + 2y &=& 49 \end{array}\right|\begin{array}{l} \\ \cdot 2 \end{array}$$

$$\left\{\begin{array}{rcl} 2x + 3y &=& 96 \\ x + 4y &=& 98 \end{array}\right. .$$

When isolating x from the second equation, we get

$$x = 98 - 4y.$$

We can therefore substitute

$$\begin{array}{rcl|l} 2(98-4y) + 3y &=& 96 & \\ 196 - 8y + 3y &=& 96 & -196 \\ -5y &=& -100 & :(-5) \\ y &=& 20. & \end{array}$$

So

$$x = 98 - 4\cdot 20 = 98 - 80 = 18.$$

The vertex is then $E(18;20)$.

To find the maximum value of P, we will calculate it for each vertex of the polygon. We obtain the following values:

Vertex $(x;y)$	Value of $P = 70x + 160y$
$A(0;0)$	$70\cdot 0 + 160\cdot 0 = 0$
$B(0;24,5)$	$70\cdot 0 + 160\cdot 24.5 = 3'920$
$C(30;0)$	$70\cdot 30 + 160\cdot 0 = 2'100$
$D(30;12)$	$70\cdot 30 + 160\cdot 12 = 4'020$
$E(18;20)$	$70\cdot 18 + 160\cdot 20 = 4'460$

Hence, the maximal profit of 4'460 francs will be obtained by producing 18 plates and 20 vases.

Example. We want to prepare food rations containing at least 90 g of protein, 120 g of carbohydrates and 2'400 calories from two products A and B. One dose of product A costs 1 franc and contains 15 g of protein, 20 g of carbohydrates and 300 calories. One dose of product B costs 1 franc and contains 10 g of protein, 30 g of carbohydrates and 400 calories. What is the composition of the most economical food ration?

The above information is grouped together in the following table.

	Protein	Carbohydrate	Calories	Price
Product A	15 g	20 g	300 g	1 Fr
Product B	10 g	30 g	400 g	1 Fr
Total	≥ 90	≥ 120	$\geq 2'400$	P

If x is the number of doses of the product A and y is the number of doses of the product B, the problem is to find the minimum value of the expression

$$C = x + y$$

with the constraints

$$\begin{cases} 15x + 10y & \geq & 90 \\ 20x + 30y & \geq & 120 \\ 300x + 400y & \geq & 2'400 \\ x & \geq & 0 \\ y & \geq & 0 \end{cases}.$$

Let's draw the lines $15x+10y = 90$, $20x+30x = 120$, $300x+400 = 2'400$, $x = 0$ and $y = 0$. Let's compute two points per line.

1. Points of $15x + 10y = 90$:

   — For the first point, we put $x = 0$:
   We have
   $$\begin{array}{rcl|l} 15 \cdot 0 + 10y & = & 90 & \\ 10y & = & 90 & : 10 \\ y & = & 9. & \end{array}$$
   So the point is $(0; 9)$.

   — For the second point, we put $y = 0$:
   We have
   $$\begin{array}{rcl|l} 15x + 10 \cdot 0 & = & 90 & \\ 15x & = & 90 & : 15 \\ x & = & 6. & \end{array}$$
   So the point is $(6; 0)$.

   Hence, the line $15x + 10y = 90$ passes through $(0; 9)$ and $(6; 0)$.

2. Points of $20x + 30y = 120$:

— For the first point, we put $x = 0$:
We have

$$\begin{array}{rcl|l} 20 \cdot 0 + 30y & = & 120 & \\ 30y & = & 120 & : 30 \\ y & = & 4. & \end{array}$$

So the point is $(0; 4)$.

— For the second point, we put $y = 0$:
We have

$$\begin{array}{rcl|l} 20x + 30 \cdot 0 & = & 120 & \\ 20x & = & 120 & : 20 \\ x & = & 6. & \end{array}$$

So the point is $(6; 0)$.

Hence, the line $20x + 30y = 120$ passes through $(0; 4)$ and $(6; 0)$.

3. Points of $300x + 400y = 2'400$:

— For the first point, we put $x = 0$:
We have

$$\begin{array}{rcl|l} 300 \cdot 0 + 400y & = & 2'400 & \\ 400y & = & 2'400 & : 400 \\ y & = & 6. & \end{array}$$

So the point is $(0; 6)$.

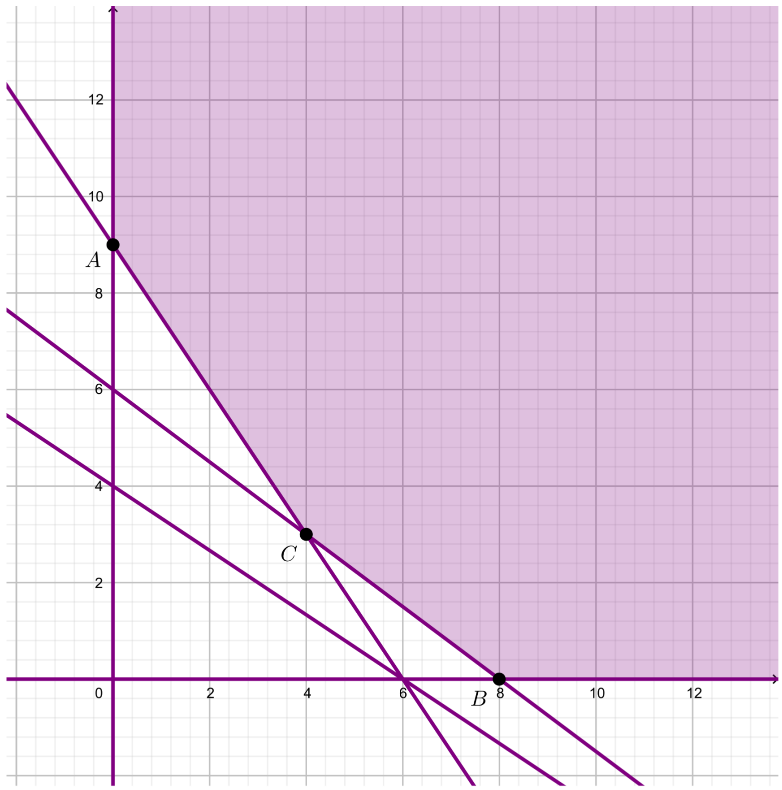
— For the second point, we put $y = 0$:
We have

$$\begin{array}{rcl|l} 300x + 400 \cdot 0 & = & 2'400 & \\ 300x & = & 2'400 & : 300 \\ x & = & 8. & \end{array}$$

So the point is $(8; 0)$.

Hence, the line $300x + 400y = 2'400$ passes through $(0; 6)$ and $(8; 0)$.

4. Points of $x = 0$ and $y = 0$:
The line $x = 0$ is a vertical line passing through the origin. The line $y = 0$ is a horizontal one passing through the origin.



The area defined by the constraints is shown above. We notice that the constraint $20x + 30y \geq 120$ is satisfied if the others are.

The minimum sought is necessarily reached on a vertex of the polygon.

Let us then determine the coordinates of these vertices:

It's clear that points $A(0,9)$ and $B(8,0)$ are vertices of the polygon.

Now we have to determine the coordinates of the last vertex C.
To do that, we solve

$$\left\{\begin{array}{rcl|l} 15x + 10y & = & 90 & :5 \\ 300x + 400y & = & 2'400 & :100 \end{array}\right.$$

$$\begin{array}{rrcl} & 3x + 2y & = & 18 \\ - & 3x + 4y & = & 24 \\ \hline & -2y & = & -6 \\ & y & = & 3. \end{array}$$

We find x by solving

$$\begin{array}{rcl|l} 3x + 2 \cdot 3 & = & 18 & \\ 3x + 6 & = & 18 & -6 \\ 3x & = & 12 & :3 \\ x & = & 4. & \end{array}$$

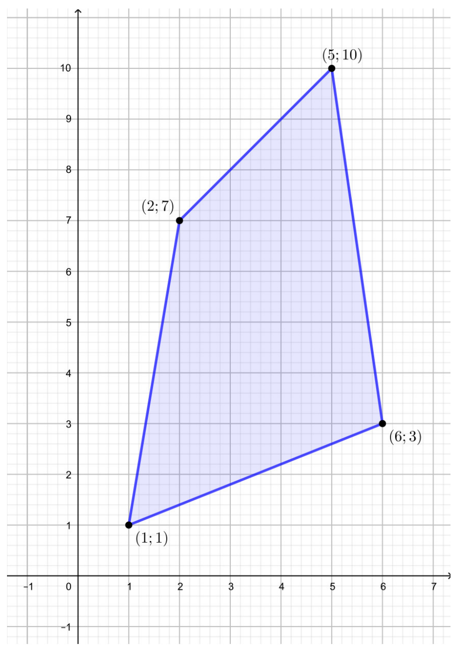
So the vertex is $C(4;3)$.

To find the minimum value of P, we will calculate for each vertex of the polygon. We obtain the following values:

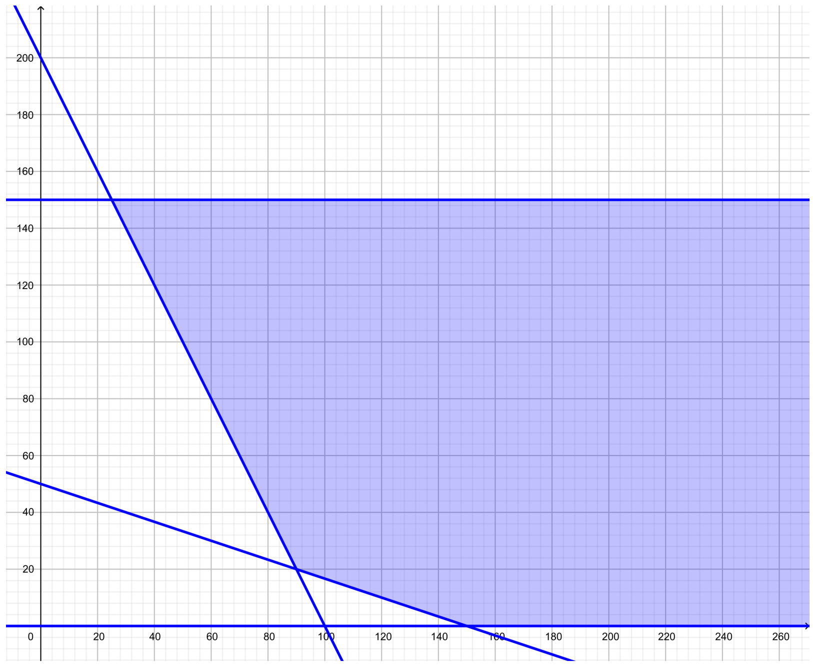
Vertex $(x; y)$	Value of $C = x + y$
$A(0; 9)$	$0 + 9 = 9$
$B(8; 0)$	$8 + 0 = 8$
$C(4; 3)$	$4 + 3 = 7$

Therefore, the minimum cost $P = 7$ francs is reached for $x = 4$ and $y = 3$. This ration contains $4 \cdot 15 + 3 \cdot 10 = 90$ g protein, $4 \cdot 20 + 3 \cdot 30 = 170$ g carbohydrates and $4 \cdot 300 + 3 \cdot 400 = 2'400$ calories.

Exercise 9.2. Determine for which point(s) of the solution below is the function $Z = 3x + 5y$ maximal and minimal.



Exercise 9.3. The shaded area is the solution set of a linear programming problem.



Answer the following questions and justify them :

a) $(50; 100)$ is a solution?

b) $(140; 60)$ can it be the optimal solution?

c) $(100; 0)$ can it be the optimal solution?

d) Consider $x = 150$; is it the equation of a border?

e) Consider $2x + y = 200$; is it the equation of a border?

Exercise 9.4. A fish farmer will buy a maximum of 5'000 young trouts and perches at a breeder's and give them special food for next year. Fish food costs 0,50 franc per trout and 0,75 franc per perch, and the cost total must not exceed 3'000 francs. At the end of the year, an ordinary trout will weigh 1,36 kg and a perch 1,80 kg. How many fish of each type should be raised in the fish tank so that the total number of kilos of fish at the end of the year is maximum?

Exercise 9.5. A small store sells computers and printers. The available room in the store is a maximum of 30 machines. On average, each computer costs 2'000 francs and each printer 800 francs. For insurance reasons, the shop does not wish to exceed 40'800 francs for the value of the stock. If its profit is 200 francs per computer and 100 francs per printer, how many computers and printers should this store have in stock to maximize his profit?

Exercise 9.6. A firm manufactures cars and trucks in a factory divided into two workshops; « workshop A » for assembling and building, and « workshop B » for all the finishing operations. Workshop A needs 5 days of work for the trucks and 2 days of work for the cars. Workshop B needs 3 days of work whether it's a car or truck. Due to limitations of staff and machinery, workshop A has a maximum of 180 working days per year and workshop B has a maximum of 135 days per year.

a) If the firm makes a profit of 3'000 francs per truck and 2'000 francs per car, how many vehicles of each type should it produce in order to maximize profits?

b) Same question for the respective profits of 4'000 francs and 1'000 francs?

Exercise 9.7. A tennis club would like to order balls and tennis rackets for juniors. It wishes to use Wilson Junior balls to for the kids and Tretorn balls for the adults. Wilson balls are packed in tubes containing 3 balls and the Tretorn in tubes containing 4 balls. The supplier A offers a set containing 3 Wilson tubes, 10 Tretorn tubes and one junior racket at the price of 200 francs. Supplier B offers a set containing 4 Wilson tubes, 6 Tretorn tubes and one junior racket at the price of 150 francs. The club would like to place an order for a certain number of sets from both suppliers. For its needs, the club wishes to have at least the stock used last year, which was 117 Wilson balls, 344 Tretorn balls and 10 junior rackets. Determine the number of sets to be ordered from the two suppliers in order to minimize costs.

Exercise 9.8. A student decides to use his brand new computer skills to earn some money. He then offers two kinds of individual courses: beginners' courses for 240 francs and advanced courses for 300 francs. Each course is divided into three parts:

— Theory, 2 hours per beginners' course and 3 hours per advanced course.

— The practice, 6 hours per beginners' course and 4 hours per advanced course..

— Individual work (his computer is available), 10 hours per beginners' course and 10 hours per advanced course.

It has set itself the following limits for the entire school year:

— Maximum 30 hours of theory

— Maximum 60 hours of practice.

— Make his computer available for a maximum of 110 hours.

How much of each of the two courses would the student be interested in organizing to earn as much money as possible while respecting all the constraints he has set for himself? How much can he earn in this way?

Exercise 9.9. You want to make a flower park, you need at least 1'200 hyacinths, 3'200 tulips and 3'000 narcissus. Two florists offer their lots:

— Lot A : 30 hyacinths, 40 tulips and 30 narcissus for 75 francs.

— Lot B : 10 hyacinths, 40 tulips and 50 narcissus for 60 francs.

Determine the number of lots of each type that must be purchased for the park with a minimal expense. Are there any flowers left for you?

Exercise 9.10. A carpentry company is specialized in the manufacture of wooden boxes. In anticipation of a big order, they decide to refill their stock. A worker produces large red boxes and another one little yellow boxes. Each red box has a volume of 20 dm^3 and each yellow box has a volume of 10 dm^3. The closet designed to store the boxes has a volume of 4'000 dm^3. For technical reasons, the first worker can only produce a maximum of 150 red boxes and the second only 200 yellow boxes. Considering that the red boxes generate a profit of 80 francs and the yellow boxes 30 francs, how many boxes of each colour should the company make to maximize his profit?

Exercise 9.11. A man proposes to install a booth at a fair to sell packets of peanuts and candies. He has 800 francs to acquire the packets, which cost 80 cents per packet of peanuts and 1,60 francs per packet of candies. The profits are 2 francs per packets of peanuts and 3,20 francs per packet of candies. His booth can hold 500 packets of peanuts and 400 packets of candy. From experience, he knows he will not sell more than 700 packets in total. Determine the number of packets of peanuts and candies that he must have in his possession in order to achieve maximum profits.

Exercise 9.12. The organizer of a tour must rent buses to transport 400 people. The organizer has chosen a transport company that has 10 large buses, which can hold 50 people and six small buses, which can hold 25 people. The first type of bus costs 2'500 francs at rent and the second costs 1'500. Determine the number of large and small buses that the organizer must rent in order to keep the cost of renting to a minimum.

Exercise 9.13. The Badler company sells calculators and notebooks. The calculators are sold for 5 francs and are manufactured for 3 francs. The notebooks are also sold at 5 francs but its production costs are 40% cheaper than the calculators ones. This company can produce from 2'000 to 4'000 calculators and from 1'000 to 5'000 notebooks. In addition, it cannot sell more than 8'000 articles in total. How many calculators and notebooks does it need to produce to maximize its profit?

Exercise 9.14. An automobile factory builds two models: A and B. Each day, it can produce a maximum of 600 cars A and 200 cars B, but due to a lack of manpower, it cannot produce more than 750 cars in total. In addition, the production of the model B cannot exceed half of the production of the model A. The profit is 1'200 francs for a car A and 1'800 francs for a car B. Determine the production that will ensure maximum profit for the company. What is this profit?

9.3 Solutions

Exercise 9.1. Yes, we need to manufacture 40 boxes A and 0 box B for a profit of 2'000 francs.

Exercise 9.2.

Maximum: $P = 65$ with $(x; y) = (5; 10)$.

Minimum: $P = 8$ with $(x; y) = (1; 1)$.

Exercise 9.3.

a) Yes.

b) No.

c) No.

d) No.

e) Yes.

Exercise 9.4. We need 3'000 trouts and 2'000 perches in order to maximize the total weight.

Exercise 9.5. The maximum profit will be obtained with 14 computers and 16 printers.

Exercise 9.6. The maximum profit will be obtained with

a) 30 trucks and 15 cars.

b) 36 trucks

Exercise 9.7. The minimum cost will be obtained with 5 sets A and 6 sets B.

Exercise 9.8. The maximum profit will be obtained with 3 beginners' courses and 8 advanced courses for a total of 3'120 francs.

Exercise 9.9. The minimal expense will be obtained using 20 lots A and 60 lots B. You have 600 narcissus left.

Exercise 9.10. The maximum profit of 15'000 francs is obtained with 150 red boxes and 100 yellow boxes.

Exercise 9.11. The maximum profit is reached with 400 packets of peanuts and 300 packet of candies.

Exercise 9.12. The minimum costs are obtained using 8 large bus.

Exercise 9.13. The maximum profit is obtained with 3'000 calculators and 5'000 notebooks.

Exercise 9.14. The maximum profit of 1'020'000 francs will be obtained with 550 cars A and 200 cars B.

9.4 Chapter objectives

At the end of this chapter, the student should be able to

9.1 □ Solve a problem of linear programming.

Chapter 10

Introduction to descriptive statistics

10.1 Introduction

Statistics comes from the Latin word *status* which means state, situation. The earliest records of statistics are the surveys that were conducted in the first centuries of our era. It wasn't until the 18th century, though, that it was established as an independent scientific discipline. Today, statistics is a branch of applied mathematics involved in various fields of human thought such as demography, economics, medicine, agronomy and industry.

The statistical approach can be divided into five stages. Firstly, the population and the character(s) to be studied must be precisely identified. Following this, data will be collected by census or sampling. Then, the data will have to be grouped, classified and presented (*descriptive statistics*). It will then be necessary to compare the results with theoretical models (*calculus of probabilities*). Finally, the results must be interpreted and plausible assumptions made for predictions (*inferential statistics*) about similar circumstances.

This chapter is limited to an introduction to descriptive statistics by presenting, on the basis of two illustrative examples, the few measures that characterize a finite data set.

10.2 Definitions

Definition.

1. We call *population* the reference set used for studies. We call N the total number of elements in the population.
2. We call *sample* a sub-set of the population (when it's too big).
3. We call *individual* a element of the population or the sample.
4. We call *statistical variable* the studied characteristic that every individuals have.
5. We call *value* or *outcome* the possible values for the variable (can also be called observation).
6. The number of individuals with one specific outcome is called *frequency.*
7. The *relative frequency* or *percent* of an outcome is the ratio of the frequency to the total number of observations. We often express it as a percentage.

Notation. We denote a statistical variable by a capital letter X, Y, ... and its outcomes by the same small letter with indexes: x_1, x_2, ... for the variable X or y_1, y_2, ... for the variable Y.

Example. We're conducting a survey among EPC's students. We would like to know the gender, age at January 1st, size, and the level of satisfaction with the studies (satisfied (S), unsatisfied (U) and no answer (NA)) of each student.

The population considered is "EPC's students". A *sample* is, for example, all students in their final year of education. Any student in the final year of education is an *individual*.

Statistical variable	Outcome
X: gender	$x_1 =$ man, $x_2 =$ woman
Y: age	$y_1 = 18$, $y_2 = 19$, ...
Z: size	$z_i \in [150; 200]$
U: level of satisfaction	$u_1 = S$, $u_2 = U$, $u_3 = NA$

Definition.

1. A statistical variable is said to be *quantitative*, respectively *qualitative* if its values can be counted, respectively if they cannot be counted.
2. A qualitative variable X is said to be *nominal* or *categorical*, respectively *ordinal*, if its values have no order, respectively if its values have some natural order.
3. A quantitative variable X is said to be *discrete*, respectively *continuous*, if its set of values is finite or infinite countable, respectively if the outcomes can take any value in a interval.

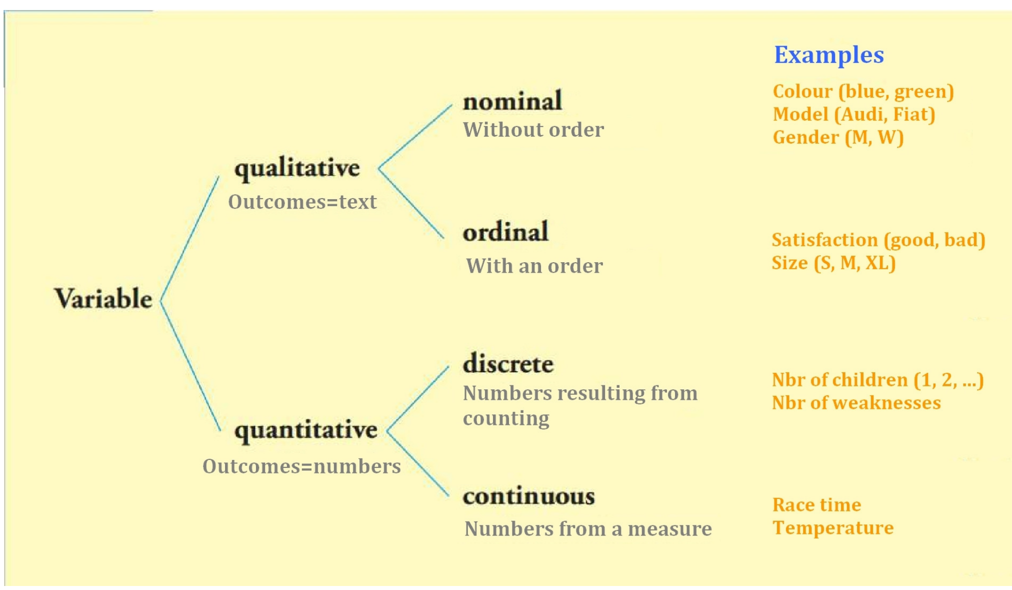


Figure 10.1: Classification of the different statistical variables.

Example. In our previous example, X is a nominal (qualitative) variable, Y is a discrete (quantitative) variable, Z is a continuous (quantitative) variable, and U is an ordinal (qualitative) variable.

Exercise 10.1. New passenger cars registered in 2015 are divided in the table below according to the type of energy consumed.

Type of energy	Thousands of cars
Petrol	1'450
Diesel	630
Hybrid	32
Other	9

a) What is the population?

b) What is the variable?

Exercise 10.2. In each of the following situations, determine whether it is a qualitative (nominal or ordinal) or quantitative (discrete or continuous) variable.

a) The size of the students attending the University of Neuchâtel.

b) The number of pages of a course document.

c) The nationality of the students in a class.

d) The weight of a newborn baby.

e) The degree of qualification of a company's staff.

f) The number of rainy days during the month of August.

g) The degree of satisfaction of CPLN students with the mathematics course.

h) The home address of the children in a elementary school class.

i) The wind speed.

j) The number of typos on each page of a newspaper.

k) The number of people living in households in the canton of Neuchâtel.

l) The seriousness of injuries of people admitted to a hospital emergency room.

m) The civil status of the inhabitants of Switzerland.

n) The number of television per family.

o) The time spent by each patient in a doctor's office.

p) The ratings of a ski resort's slopes.

Exercise 10.3. Employees of a company were asked which political party they voted for in the last election. Here is the raw data obtained:

PS	PLR	PS	PDC	PS	UDC
PS	UDC	PLR	PS	verts	PDC
UDC	PLR	verts	UDC	UDC	UDC
PLR	PS	PLR	PDC	PLR	PDC
UDC	PDC	PS	UDC	UDC	UDC

a) Identify the population and the variable.

b) List all the outcomes.

c) What type of variable is this?

Exercise 10.4. A university teacher recorded the number of points obtained by 80 students in a statistics test.

2	3	5	5	4	6	6	5	4	3
7	7	7	6	2	7	7	9	8	10
5	6	6	8	6	6	3	7	3	5
9	7	6	4	7	5	9	9	6	9
6	3	9	8	8	7	5	6	10	6
9	7	7	7	4	7	10	8	7	10
3	5	8	5	8	7	4	8	10	7
4	6	6	8	7	7	7	8	8	9

a) Identify the population and the variable.

b) Give all the outcomes.

c) What type of variable is this?

10.3 Data processing

10.3.1 Grouping data by outcomes

Example. We're looking at the civil status of a company's 40 employees.

The first step is to collect the information, in this case the civil status of each individual in the population (the 40 employees of the company): *the raw data.* The statistical variable is the civil status. It is a nominal variable and the outcomes are: married, single, divorced and widowed.

The civil status of employees identified by a number is given:

1	Married	11	Widowed	21	Single	31	Single
2	Married	12	Married	22	Married	32	Divorced
3	Single	13	Single	23	Married	33	Divorced
4	Divorced	14	Single	24	Married	34	Married
5	Married	15	Married	25	Divorced	35	Married
6	Single	16	Single	26	Married	36	Married
7	Single	17	Married	27	Single	37	Married
8	Married	18	Widowed	28	Single	38	Married
9	Married	19	Married	29	Married	39	Single
10	Divorced	20	Divorced	30	Widowed	40	Married

This raw data is used to obtain personalised information. The individual character of the information will be sacrificed in order to obtain an overall picture. We calculate for each outcome the number of individuals having this outcome. *It's the frequency of the outcome*:

20 married individuals
11 single individuals
6 divorced individuals
3 widowed individuals

It is usual to present the distribution of frequency in the form of a table:

Outcome	Frequency	Percent
Married	20	50%
Single	11	27,5%
Divorced	6	15%
Widowed	3	7,5%
Total	40	100%

In order to find that we have 27,5% of single employees, we need to compute

$$\frac{11}{40} = 0,275 = 27,5\%.$$

Example. In a neighbourhood composed of 50 households, the number of persons per household is studied.

The first step is to collect raw data on each individual in the population (the 50 households). The statistical variable is the number of persons per household. It is a discrete variable and the outcomes are: 1, 2, 3, 4, 5, 6 and 8.

The raw data are:

1	1	1	1	1	2	2	2	2	2
2	2	2	2	3	3	3	3	3	3
3	3	3	3	3	3	3	3	3	4
4	4	4	4	4	4	4	4	4	5
5	5	5	5	5	6	6	6	8	8

This raw data is used to obtain personalised information. As this list is not easy to read, the individual character of the information must again be sacrificed to obtain an overall picture. We will therefore determine for each outcome the number of individuals having this outcome. *It's the frequency of the outcome.*

Outcome x_i	Frequency n_i	Percent f_i
1	5	10%
2	9	18%
3	15	30%
4	10	20%
5	6	12%
6	3	6%
7	0	0%
8	2	4%

In the table above, we have $x_1 = 1$, $x_2 = 2$, etc. The x_i represents the number of persons per household. In addition, we have $n_1 = 5$, $n_2 = 9$, etc. The n_i indicates the number of households with x_i persons. For example, we have 10 households with 4 persons.

Note also that the penultimate line counting 0 households with 7 persons is unnecessary and that the value 7 is not a outcome of the variable.

10.3.2 Representation of data with classes

Often, in a statistical survey of a discrete or continuous variable, the data collected differ from one another and are spread over a wide range of values. Since the objective of descriptive statistics is to summarize this data set as adequately as possible, we proceed to a grouping of these in *classes*, that is to say, sub-intervals of values. The following rules allow a judicious choice of these classes:

- The number of classes is set between 5 and 15. The number of classes depend on the size of the population and frequencies of classes that are too small should be avoided if possible.
- The intervals are of the type $[b_{i-1}; b_i[$ or $]b_{i-1}; b_i]$.

  b_{i-1} is the *lower bound* of the class i;
  b_i is the *upper bound* of the class i;
  $c_i = \dfrac{b_{i-1} + b_i}{2}$ is the *centre* of the class i;
  $L_i = b_i - b_{i-1}$ is *the length* (or *the size* or *the width*) of the class i.

- In theory, the bounds of the intervals are set so that they are of equal length. The bounds should enable simple calculations to be made.
- If you really need to use classes of unequal sizes, put the classes of equal size in the centre of the distribution.

Example. In a French region, the surface area of each of the 500 farms expressed in hectares is studied.

In this example, the population is the total number of farms in a French region, while an individual is a given farm. The population being defined, it is observed according to certain criteria. The criterion used, i.e. the statistical variable, is the area. It is a continuous variable and the outcomes are numbers representing areas between 0 ha and 40 ha.

The raw data collected on this population cannot be used unchanged. In order to summarize the information, groupings, classifications and statistical tables are made. The table below is already a first simplification of the complete information contained in an official register with one line for each of the 500 farms.

Area in ha	Number of farms	Percent in %
$]0; 10]$	48	$9,6$
$]10; 15]$	62	$12,4$
$]15; 20]$	107	$21,4$
$]20; 25]$	133	$26,6$
$]25; 30]$	84	$16,8$
$]30; 40]$	66	$13,2$

Since the individuals are grouped in *classes*, we're dealing with a *grouped* set of data. What's gained in simplicity through grouping is lost in information. We know, for example, that the class $]20;25]$ has 133 farms with areas between 20 and 25 ha. However, we know nothing about the distribution of these 133 individuals within their class. It is then convenient to assume a uniform distribution within each class. The individual occupying the x place out of 133 in the class $]20;25]$ (of *length* 5) is therefore assigned to the value $20 + \frac{x}{133} \cdot 5$. With this convention, the last individual (the 133^{rd}) is assigned to the value 25, upper bound of the interval.

Exercise 10.5. Chemists have just composed a new synthetic fibre which should be characterized by its resistance. To check its tensile strength, a random sample of 60 fibres is taken from production and tested. The results (in kg) are as follows:

35	65	71	75	77	80	81	82	84	86	87	89	91	97	100
48	69	72	75	78	80	81	83	85	86	88	89	94	97	103
53	69	73	76	79	80	81	83	85	87	88	89	95	99	104
63	71	74	77	79	81	82	84	86	87	89	91	97	99	114

Group the data into 6 classes of length 15 with 30 as a minimum value. For each class, give the mid value, the corresponding frequencies and the relative frequencies.

10.4 Graphical representations

10.4.1 Pie chart

The distribution of a population and its percent distribution is sometimes more visually meaningful when represented using a *pie chart*. A pie chart involves representing the total population by a disc and dividing it into slices in proportion to the number of people in the statistical variable of interest. This provides a graphical representation of the relative distribution of the population, i.e. the percent distribution.

Example. Let us take the example of farms again. What defines "the size of a slice" is the angle at the centre. To find it, simply make a rule of three with the relation 360° corresponds to a percent of 100% or, equivalently, to a frequency of 500.

Area in ha	Frequency	Percent in %	Angles in °
$]0;10]$	48	9,6	34,56
$]10;15]$	62	12,4	44,64
$]15;20]$	107	21,4	77,04
$]20;25]$	133	26,6	95,76
$]25;30]$	84	16,8	60,48
$]30;40]$	66	13,2	47,52

Figure 10.2: Data with angles.

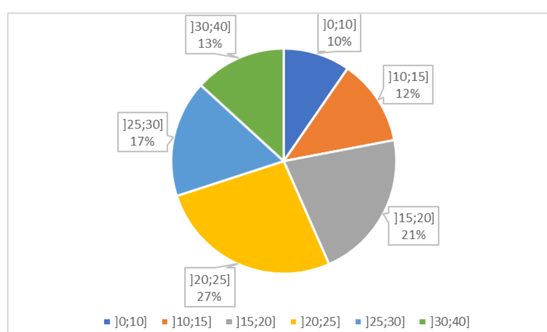


Figure 10.3: Pie chart.

Example. Let us take our example of the civil status of a company's employees. To represent the pie chart, the angle of each slice must be determined.

Civil status	Frequency	Percent in %	Angles in °
Married	20	50	180
Single	11	27, 5	99
Divorced	6	15	54
Widowed	3	7, 5	27

Figure 10.4: Data with angles.

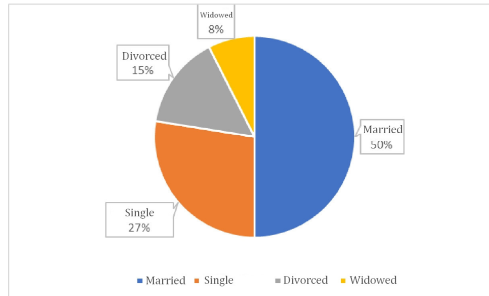
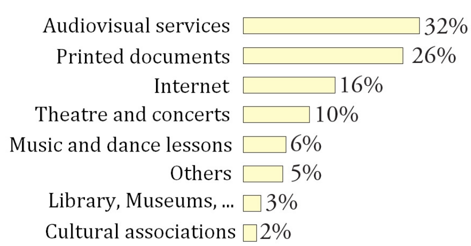


Figure 10.5: Pie chart.

Exercise 10.6. From 2009 to 2011, cultural spending represented 5% of the total Swiss household spending. Their distribution is given in the graph below.



Determine whether the following statements are true or false. Justify.

a) The studied population is "Swiss households".

b) The character studied is "categories of cultural spending".

c) It's a discrete variable.

d) This diagram is wrong, because the sum of the percentages should be 5%.

Exercise 10.7. A survey was conducted on 1'400 teenage girls who had attended a concert by the group *One Direction*. The question asked was, "Who is your favorite singer in the band?" The answers have been gathered in the table below.

Singer	Frequency	Percent (in %)	Angle (in degrees)
Liam Payne	350		
Niall Horan	210		
Zayn Malik	140		
Harry Styles	560		
Louis Tomlinson	140		

a) Fill in the table.

b) Make a pie chart.

Exercise 10.8. In a survey, prior to a popular vote, the following information was obtained:

Opinion	Number
Yes	1'240
Rather yes	350
Undecided	780
Rather no	410
No	1'120

a) Determine the relative frequency (in %).

b) Draw a pie chart.

10.4.2 Bar chart

When the statistical variable is discrete, the distribution of numbers can be represented visually by a *bar graph*. This is a diagram in which the outcomes lie on the horizontal axis and each bar rises to the level of the corresponding frequency (or percent).

Example. Let us take our example of the civil status of a company's employees. The bar chart of this distribution is shown below.

Outcomes	Frequency
Married	20
Single	11
Divorced	6
Widowed	3
Total	40

Figure 10.6: Data.

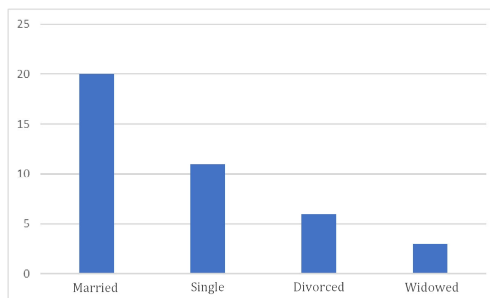


Figure 10.7: Bar chart.

Example. Let us take again our example relating to the number of persons per household.

Outcomes	Frequency
1	5
2	9
3	15
4	10
5	6
6	3
7	0
8	2

Figure 10.8: Data.

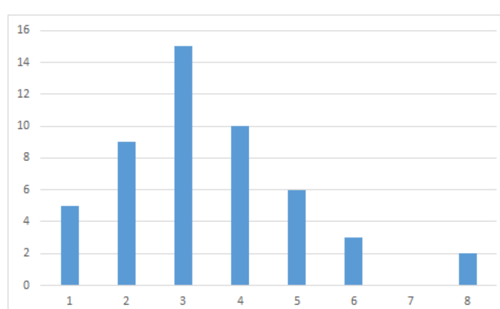


Figure 10.9: Bar chart.

Exercise 10.9. A fashion store is interested in the colour of the dresses in its collection. By recording the colour of all its dresses, the raw data below was collected.

yellow	green	yellow	black	pink	black
blue	blue	white	orange	green	beige
red	red	white	red	orange	blue
blue	pink	beige	green	white	yellow
green	green	beige	white	red	pink
black	yellow	pink	pink	blue	beige
white	orange	green	grey	beige	green
black	beige	grey	grey	black	blue
blue	black	red	black	red	red
green	grey	black	orange	green	black

a) Identify the population and the variable.

b) List all the outcomes.

c) Create a table with the frequencies and the relative frequencies.

d) Represent the data using a bar chart.

Exercise 10.10. We study the civil status of the 30 employees (numbered from 1 to 30) of a small company.

1	Married	11	Married	21	Single
2	Married	12	Single	22	Married
3	Single	13	Married	23	Widowed
4	Divorced	14	Widowed	24	Single
5	Married	15	Married	25	Divorced
6	Single	16	Divorced	26	Divorced
7	Single	17	Single	27	Married
8	Married	18	Married	28	Married
9	Married	19	Married	29	Married
10	Divorced	20	Married	30	Married

a) Identify the population.

b) Identify the variable.

c) Give all the outcomes.

d) Create the table with the frequencies and the percents.

e) Sketch the frequency bar chart.

f) Sketch the percent bar chart.

g) Compare these two graphical representations.

10.4.3 Histogram

When the statistical variable is continuous or discrete and the data are grouped into classes, the distribution can be represented by a *histogram*, which is a bar chart where the rectangles are juxtaposed. Indeed, the outcomes are here replaced by classes and these are formed of successive intervals so that there is no longer any need to separate these rectangles.

Example. In our example of farms, the classes don't all have the same size. Some classes have a size of 10 ha, others 5 ha. To be correct, a graphical representation must take these differences into account. If, in a *histogram*, the classes are represented by rectangles, then, with the total area representing the entire population, each rectangle must have an area proportional to the size of the class it represents.

Surface in ha	Number of farms
$]0;10]$	48
$]10;15]$	62
$]15;20]$	107
$]20;25]$	133
$]25;30]$	84
$]30;40]$	66

Figure 10.10: Original frequencies.

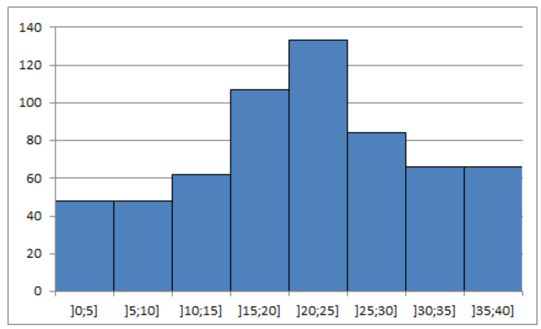


Figure 10.11: Misleading histogram.

The histogram shown above is misleading since it gives the false impression that the initial class $[0;10[$ contains 96 farms: 48 with an area from 0 to 5 ha and the same number with an area from 5 to 10 ha. To avoid this bias, a reference size (for example, 5 ha) should be chosen and the following procedure should be followed by a frequency adjustment.

With this correction, the following table and corresponding histogram are then obtained.

Surface in ha	Number of farms
$]0;5]$	$\boxed{24}$
$]5;10]$	$\boxed{24}$
$]10;15]$	62
$]15;20]$	107
$]20;25]$	133
$]25;30]$	84
$]30;35]$	$\boxed{33}$
$]35;40]$	$\boxed{33}$

Figure 10.12: Adjusted frequencies.

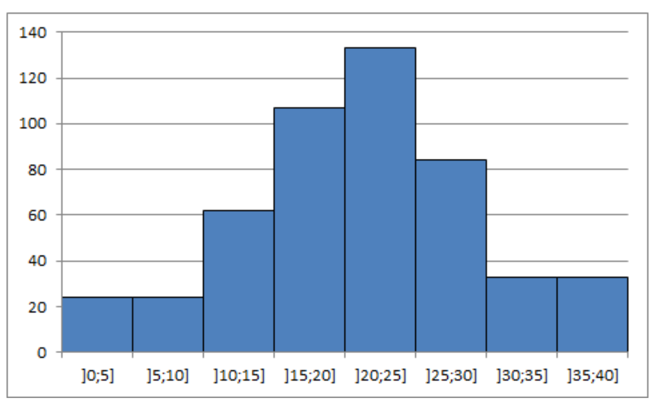


Figure 10.13: correct histogram.

Frequencies correction algorithm

1. We choose a reference class of width l (usually the most frequent one).
2. For any class of length L and whose frequency is E, we calculate the ratio $x = \dfrac{E}{L}$.
3. We then assign to this class the corrected number $c = x \cdot l = \dfrac{E}{L} \cdot l$. Let us note that this number is not necessarily an integer.

Example. In our example, with the reference class whose length is $l = 5$, the class $]0;10]$ has a length of $L = 10$, we calculate $x = \dfrac{E}{L} = \dfrac{48}{10}$ =4,8, which leads to the corrected frequency $c = x \cdot l = 4,8 \cdot 5 = 24$.

Exercise 10.11. An agricultural survey in 2015 classified farms according to the surface used for agriculture. The results are presented below.

Surface used in ha	Frequency
$[0;20[$	125’000
$[20;50[$	44’000
$[50;100[$	62’000
$[100;200[$	35’000
$[200;1000[$	12’000

a) Determine the population and the variable.

b) How do we call the intervals in the first column.

c) Why did we group the data?

d) What type of variable is this?

e) What graphical representation could be used?

Exercise 10.12. Build correctly the corresponding histogram based on the following classes and frequencies.

Classes	Frequency
$[50;100[$	3
$[100;125[$	5
$[125;150[$	4
$[150;175[$	6
$[175;200[$	5
$[200;300[$	2

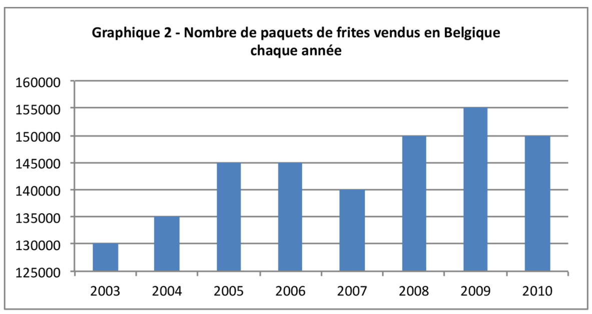
10.4.4 Misleading diagrams or fakes

In the newspapers, on television or in political leaflets, we sometimes find diagrams or graphs misrepresenting reality, or even completely false. The purpose of this section is to give some examples and to highlight the techniques used to distort reality.

Example.

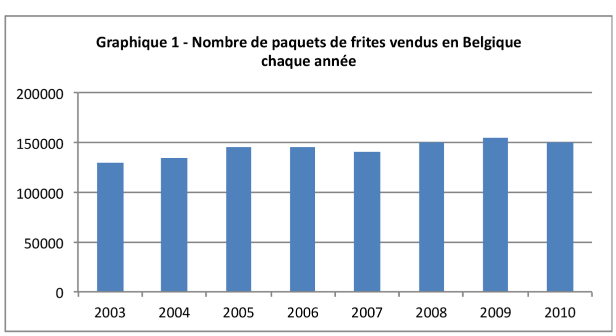
1. In this example relating to the evolution of the number of packets of French fries sold in Belgium, we will see how to present the data in order to give three radically different messages.

   a) Despite the 2007 data, the bar graph below seems to indicate an increase in sales of the number of packets of French fries.



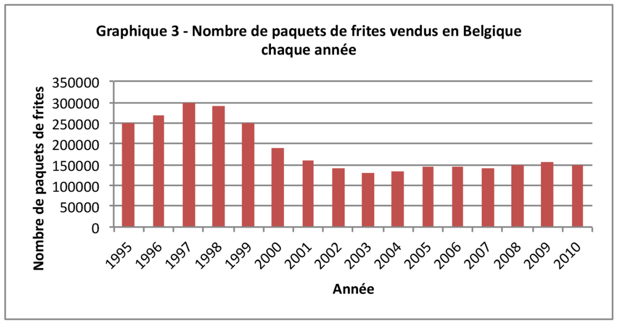
   However, a closer look reveals that the y axis does not start at 0, but at 125'000! On this diagram, we can read that 130'000 packs of french fries were sold in 2003, against 135'000 in 2004, which corresponds to an increase of 5'000 packs in one year, or about $3,85\%$. Well, the visual effect of the diagram suggests at first glance that sales doubled during this period, i.e. they increased by 100%! Notice finally that the diagram gives the impression that sales were multiplied by 6 between 2003 and 2009, while they went from 130'000 to 155'000, which is 25'000 more, an increase of only 20%!

   b) Presenting the same data, but starting the y axis from the origin, the bar chart below seems to indicate a fairly stable trend in French fry packets sales.



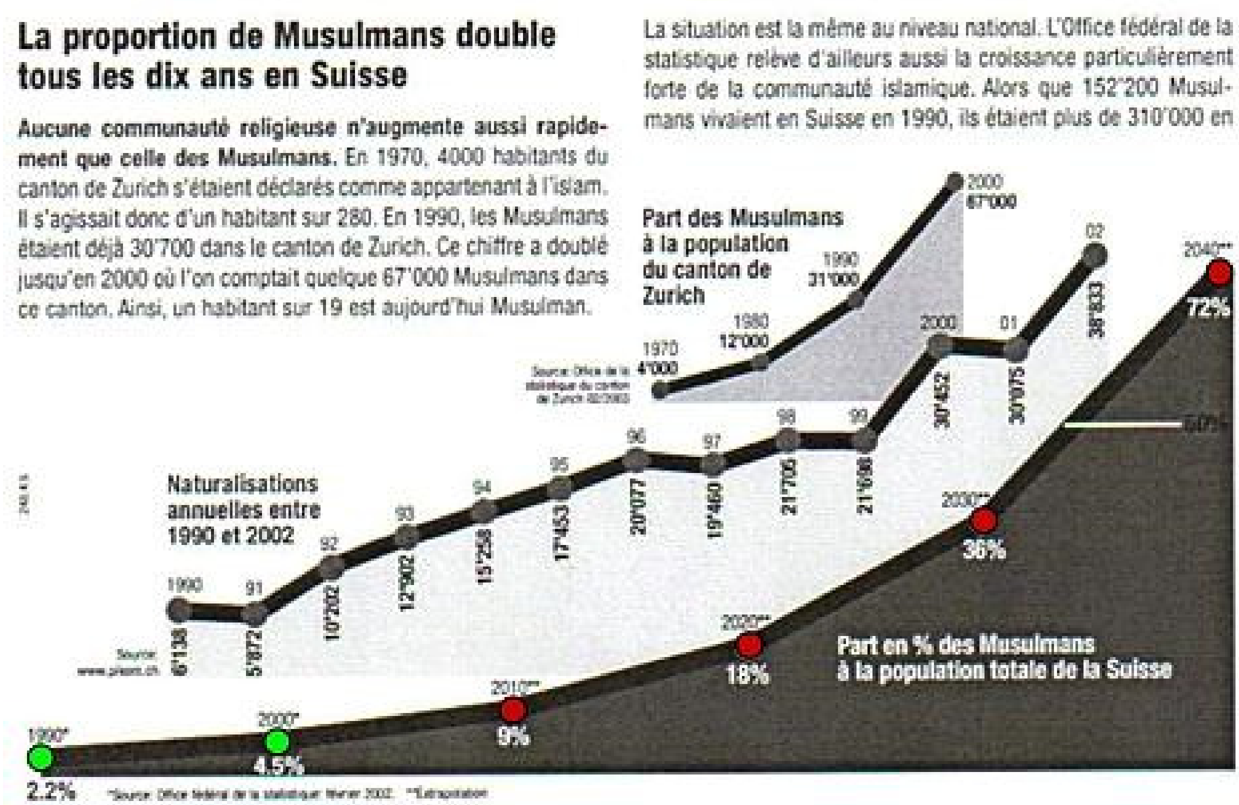
c) The two diagrams above contain only the sales data from 2003 to 2010. What if we look at the values for the years prior to 2003?

The diagram below shows the evolution of the number of packets of French fries between 1995 and 2010. Taking these data into account, it seems that the sales of French fry packets are on the decline!



With the same study, it is therefore possible to convey three completely different messages depending on how the information is presented.

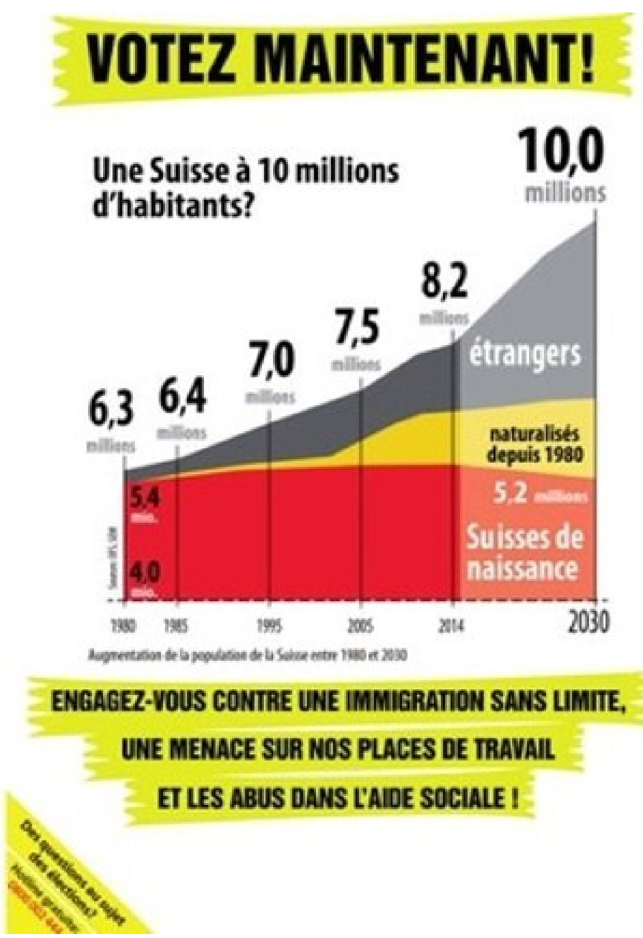
2. In view of the votes on 26 September 2004, a committee close to a political party is publishing the following document.



The text explains that no other religious community is growing as fast as Muslims. Indeed, the above curve seems to indicate that the growth in the number of Muslims in Switzerland is exponential. However, a closer look reveals that the numbers for 1990 and 2000 ($2,2\%$ and $4,5\%$) are marked with an asterisk indicating that they come from the Federal Statistical Office. The following figures (from 2010 onwards) are marked with two asterisks to indicate that this is an extrapolation.

But how can such a prediction be made? $4,5\%$ is about twice as much as $2,2\%$. With only these two values, we conclude that the percentage of the Muslim community in Switzerland doubles every 10 years to reach 72% in 2040, which is the last point represented on the graph. It is easier to understand why the graph stops at this point. Indeed, the next point would indicate that the rate of Muslims would rise to 144% in 2050! This projection is therefore based on an arbitrary increase, but reinforced by the Zurich data, which show a strong increase between 1970 and 2000. This extrapolation based on a single canton gives no reason to believe that these figures can be transposed to the whole country. Finally, it should be noted that, according to the FSO, there were $4,9\%$ Muslims in Switzerland in 2011 and $5,3\%$ percent in 2018. Those values are very different from the 9% and 18% predicted by the authors of the document above!

3. As for the poster below, it contains a number of highly questionable elements. Jörg Mäder, Zurich National Councillor since 2019, dissects the many controversial elements of this poster on this video.



4. The graph published by a daily newspaper in August 2008 (below, left) seems to show that meat consumption has stabilized in recent years. However, a closer look reveals that the horizontal axis of the graph is not linear: half of the graph represents 50 years, while the other half (the stable part) represents only 7 years, thus giving a wrong impression of the situation. The graph on the right shows the same data correctly, and gives a different impression.

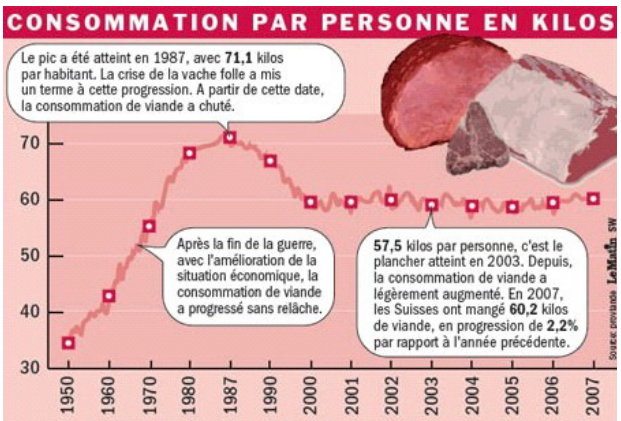


Figure 10.14: Wrong graph.

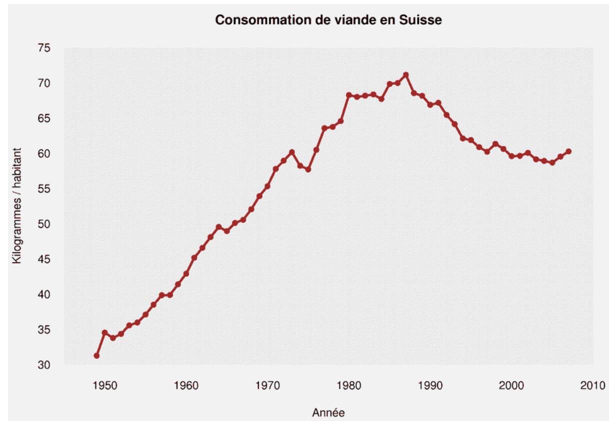
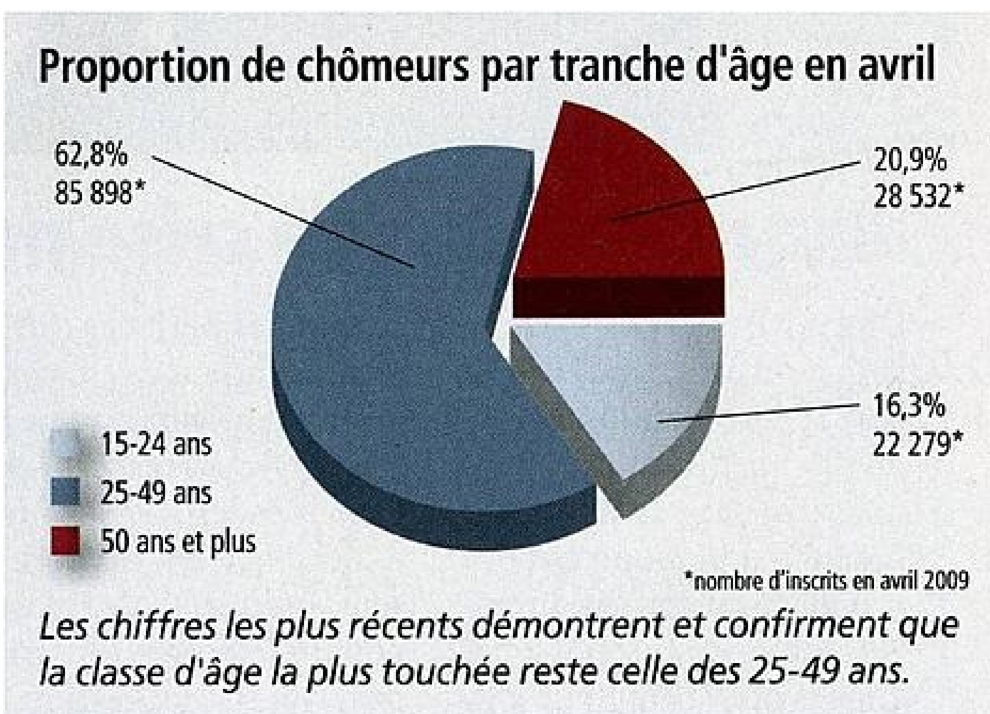


Figure 10.15: Correct graph.

Another difference between the two graphs can be seen: the one on the left shows variations within the years. In fact, it appears that these variations have been added to prevent the graph from being too smooth. It is surprising that such purely aesthetic considerations take over from the correct processing of the information.

5. The pie chart below shows the proportion of unemployed people by age group.



Caption concludes that the most affected class is the 25 to 49 year olds. Since the three classes are of different sizes, it is difficult to make comparisons. It is not surprising that the largest number of unemployed people are in the most populated class!

In fact, the interesting value is not the absolute value, but the percentage of unemployed within each class. According to the SECO report used to create the graph, these rates are 4% for 15-24 year olds, 3,6% for 25-49 year olds, and below 3% for 50-65 year olds. So the class that is most affected is the youth class, contrary to what the author of the chart says.

6. One television channel presented the diagram below in 2011. It reports the rate of public spending in 2011 in % of the GDP of three countries and the European Union.

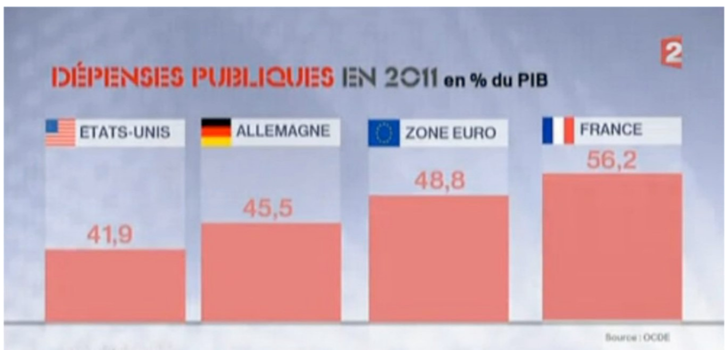


Figure 10.16: Wrong diagram.

If the 100% corresponds to the area between the horizontal and the parallel passing just below the names of the countries, the 41,9% of public spending of the United States seems to be correctly represented. This is not the case for other countries! Indeed, France's 56,2% is way too high. The chart gives the impression that France's public spending is around 80%! This technique is intended to arouse emotion among the population, by exaggerating the difference in the rate of public spending compared to other countries.

Below is the correct diagram, as it should have been presented to viewers.

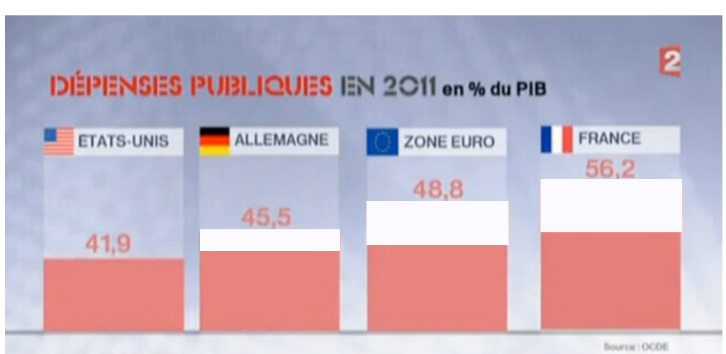
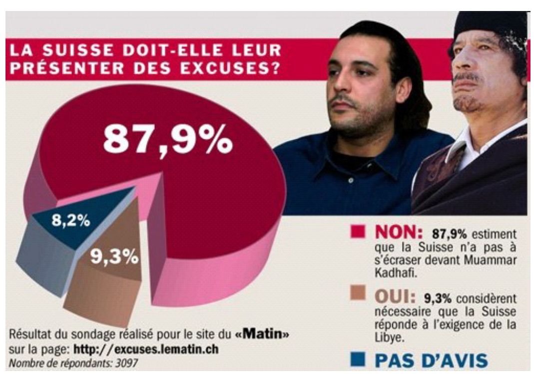


Figure 10.17: Correct diagram.

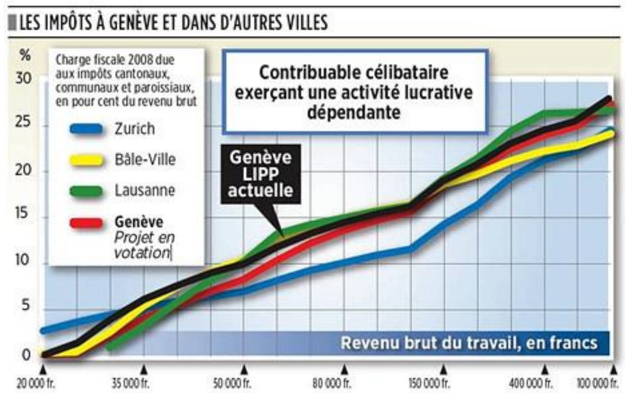
7. The figure below illustrates the fact that the price of watches has increased by 40% over 7 years.

   The graphic designer wanted to associate this increase with the diameter of the watches. We can verify that they do increase by 40%. The reader sees the increase in the surface of the clocks, which is not 40%, but close to 100%! Finally, the hands have been added for purely aesthetic purposes, but can be misleading by giving the impression that they contain information.

8. In the pie chart below, the sum of the parts is $105,4\%$ The $8,2\%$ was probably originally a $2,8\%$, which would make the expected 100%.

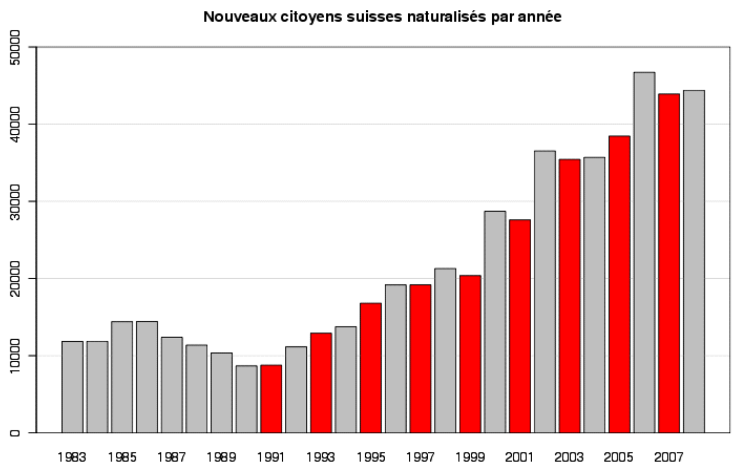


Exercise 10.13. The figure below appeared in a Geneva daily newspaper.

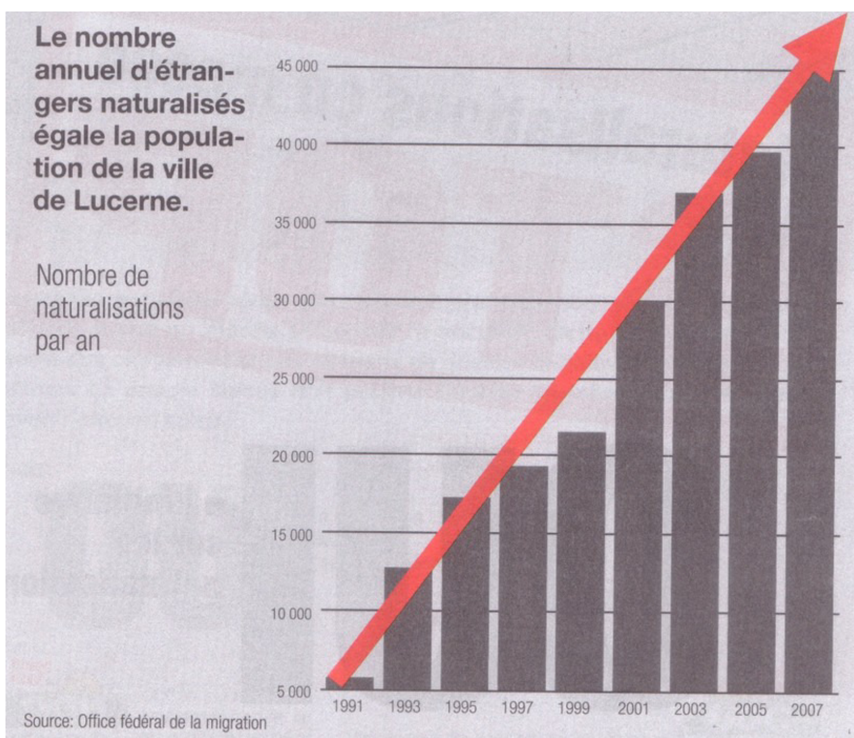


Explain how this graph is misleading.

Exercise 10.14. The diagram below shows the number of naturalized Swiss citizens per year from 1983 to 2008.



Give two techniques used to construct the following diagram.



Exercise 10.15. The figure below from 2010 explains how to correctly count the proportion of foreigners in Switzerland.

Voilà comment on compte correctement:

Si on ajoute les clandestins, les frontaliers et les requérants d'asile aux étrangers recensés officiellement, la proportion d'étrangers vivant en Suisse augmente sensiblement.

Exemple 2008:

• étrangers officiellement recensés	1 638 949
• sans-papiers (estimation moyenne)	200 000
• frontaliers	212 566
• requérants d'asile	40 797
Total des étrangers en Suisse:	**2 092 309**
	=27,2%

Proportion d'étrangers sans tenir compte des naturalisations de ces 25 dernières années: **=34,3%**

The above numbers are based on a population estimate of 7,7 million and 550'000 persons naturalized between 1985 and 2010.

a) Determine, in %, the rate of foreigners in Switzerland in 2010, considering also undocumented persons, cross-border commuters and asylum seekers.

b) How did they get the value of $27,2\%$?.

c) Explain how the number $34,3\%$ was obtained and make a judgment about its relevance.

10.4.5 Frequency polygon

We often associate to the histogram the *frequency polygon*. It is a polygonal curve such as the surface contained between this curve and the x-axis is equal to the surface of the histogram. It is a polygonal curve such as the surface contained between this curve and the axis of abscissa is equal to the surface of the histogram. It is obtained by joining the midpoints of the tops of the rectangles of the histogram. For the first and the last class, two fictitious classes of frequency zero are created for this purpose.

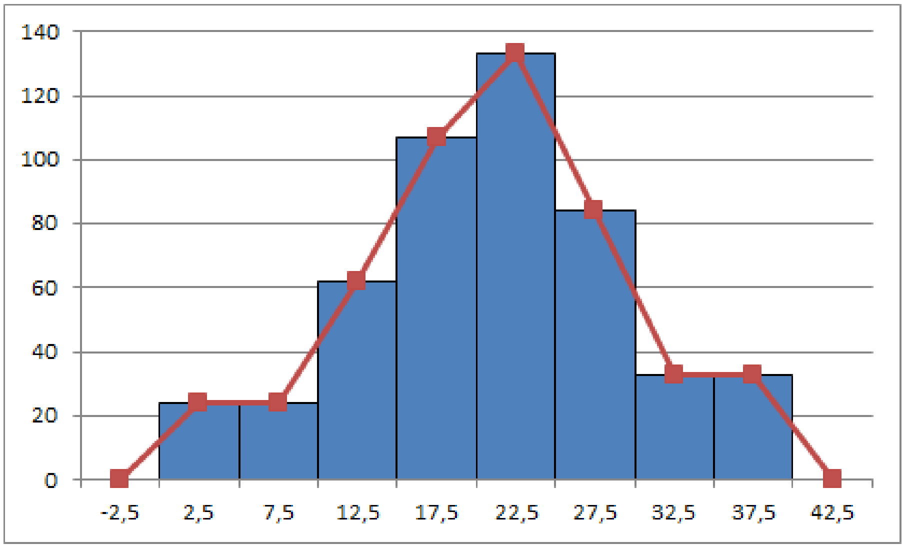


Figure 10.18: Frequency polygon.

Exercise 10.16. One company recorded the salaries of all its salesmen for the last year. Here is the data:

Classes (salaries)	Mid values	Frequency	Percent
[10000; 15000[	12500	2	
[15000; 20000[		8	10%
[20000; 25000[	22500	14	
[25000; 30000[	27500	21	26, 25%
[30000; 35000[			20%
[35000; 40000[	37500	12	15%
[40000; 45000[	42500	5	6, 25%
[45000; 50000[	47500		2, 5%
Total		80	100%

a) Fill in the table.

b) Draw a histogram.

c) Sketch the frequency polygon.

10.4.6 Cumulative frequency polygon

To the starting data, the increasing cumulative and decreasing cumulative frequency table are associated. The data in this table is understood as follows. It can be stated, for example, that 350 farms have an area of 25 ha or less. On the other hand, 283 farms have an area greater than 20 ha.

Classes	Frequency	Increasing cumulative frequency	Decreasing cumulative frequency
$]0;10]$	48	48	500
$]10;15]$	62	110	452
$]15;20]$	107	217	390
$]20;25]$	133	350	283
$]25;30]$	84	434	150
$]30;40]$	66	500	66

The data contained in this table can be represented by two curves: *increasing cumulative frequency polygon* and the *decreasing cumulative frequency polygon*. In the representation of these curves, we don't worry about differences in class size. Note that it is also possible to make a relative frequency polygon.

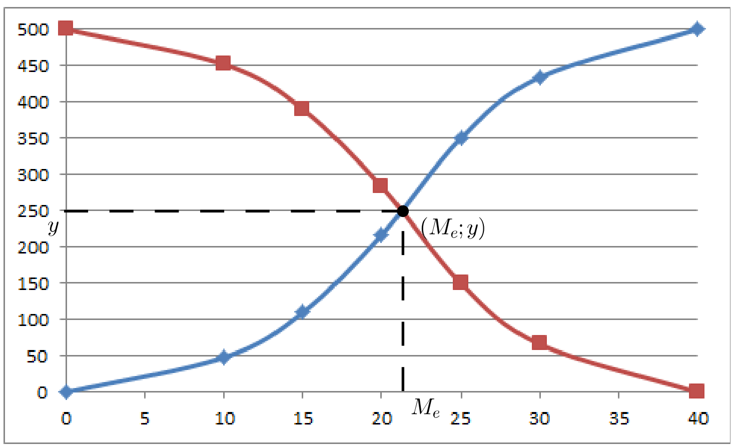


Figure 10.19: Cumulative frequency polygon.

Remark. The intersection of these two polygons is a point $(M_e; y)$, the first coordinate is called the *median* M_e of the population. We observe, in our example, that $M_e \cong 21$. This value divides the population into two parts of equal size. Indeed, that is y, the second coordinate of the point of intersection. Since this point is on the polygon of increasing cumulative frequency, we can say that y farms have an area less than M_e; the rest, i.e. $500-y$, have an area greater than M_e. Since this point is also on the decreasing cumulative polygon, y describes the number of farms with an area greater than M_e. We deduce that $y = 500-y$ and therefore, that $y = \dfrac{500}{2} = 250$. Thus half of the farms have an area greater (respectively less) than $M_e \cong 21$ ha.

Remark. In a statistical survey, if we want to know the proportion of each value that the statistical variable under consideration can take, we look at its relative frequency f_i.

If, on the other hand, we want to know the proportion of individuals with values below a fixed value, we look at the increasing cumulative frequency F_i.

To visualize the proportion of individuals who have values greater than or equal to a fixed value, we will then look at the decreasing cumulative frequency F'_i.

Exercise 10.17. The police of Fribourg recorded the following speeds on the highway on a Saturday evening.

Speed (in km/h)	Frequency
$]80; 100]$	4
$]100; 120]$	34
$]120; 140]$	84
$]140; 160]$	58
$]160; 180]$	20
Total	200

a) What type of variable is this?

b) Create the histogram of the relative frequency and the relative frequency polygon.

c) Create the cumulative relative frequency polygon.

d) How many drivers were driving too fast?

e) How many drivers will have a fine, knowing that a 4% deduction is made from the actual measured speed?

10.5 Measures of central tendency

A *central tendency* is a typical or representative value of a dataset. If this value tends to be in the middle of a dataset arranged in ascending order, then it's said to be a*measure of central tendency* or a *central tendency.*

10.5.1 Arithmetic mean

Discrete variable

Definition. The *arithmetic mean* $\overline{x}$ is the most well-known central tendency. It is equal to the quotient of the sum of all observed values of the variable by the total number. So

$$\boxed{\overline{x} = \frac{n_1 x_1 + n_2 x_2 + \ldots + n_k x_k}{N}, \qquad \text{where } N = n_1 + n_2 + \ldots + n_k}$$

Example. Let us take again our example about the number of persons per household:

Outcomes x_i	Frequency n_i	Percent f_i
1	5	10%
2	9	18%
3	15	30%
4	10	20%
5	6	12%
6	3	6%
7	0	0%
8	2	4%

The arithmetic mean $\overline{x}$ is given by

$$\overline{x} = \frac{5 \cdot 1 + 9 \cdot 2 + 15 \cdot 3 + 10 \cdot 4 + 6 \cdot 5 + 3 \cdot 6 + 2 \cdot 8}{50} = 3,44.$$

Exercise 10.18. Compute the arithmetic mean of the dataset

$$E = \{2, 3, 3, 3, 3, 4, 5, 7, 9, 10, 11, 11, 13\}.$$

Continuous variable

For grouped data sets, based on a uniform distribution within classes, all individuals in a class $]b_{i-1}, b_i]$ should be attributed to the mid value

$$c = \frac{b_{i-1} + b_i}{2}$$

.

Example. For our example with the farms, using the following table:

Classes x_i	Mid values c_i	Frequency n_i
$]0;10]$	5	48
$]10;15]$	$12,5$	62
$]15;20]$	$17,5$	107
$]20;25]$	$22,5$	133
$]25;30]$	$27,5$	84
$]30;40]$	35	66
Total		500

we deduce the arithmetic mean of the area of those 500 farms by calculating

$$\overline{x} = \frac{5 \cdot 48 + 12,5 \cdot 62 + 17,5 \cdot 107 + 22,5 \cdot 133 + 27,5 \cdot 84 + 35 \cdot 66}{500} = \frac{10500}{500} = 21 \text{ ha}.$$

Exercise 10.19. Compute the arithmetic mean of the following populations.

Ages	Frequency
$]0;20]$	72
$]20;65]$	180
$]65;100]$	43

Salary (in francs)	Number of employees
]0;80]	32
]80;100]	48
]100;260]	20

10.5.2 Mode

Example. In a village of 500 inhabitants, there are 490 people with black hair and 10 with blond hair. How can we describe the "average" hair colour of the inhabitants of this village? The answer will surely be "black", thinking that the overwhelming majority of the inhabitants have black hair. Thinking this way, we give as an answer the value that appears the most frequently. This is the *mode*.

Discrete variable

Definition. The *mode*, denoted by M_o, is the value of the variable that corresponds to the largest number or the most important frequency. This central tendency is simple to understand, but it doesn't take into account all the values of the variable studied.

Example. The numbers 3, 5, 7, 7, 7, 9, 9 have a the mode $M_o = 7$. Let's notice that it could be non-unique. Hence, the dataset 3, 5, 7, 7, 7, 9, 9, 9, which has two modes: 7 and 9, is said *bimodal*.

Example. Let's take back our example with the number of persons per household.

Outcomes x_i	Frequency n_i
1	5
2	9
3	15
4	10
5	6
6	3
7	0
8	2

The mode is given by $M_o = 3$, because 3 is the outcome with the greatest frequency.

Exercise 10.20. Indicate the mode for both data sets.

a) 7 , 8 , 9 , 3 , 3 , 7 , 6 , 7 , 8 , 7 , 3 , 9 , 6 , 7

b) 4 , 4 , 7 , 6 , 8 , 12 , 6 , 4 , 8 , 7 , 8 , 13 , 4 , 8 , 6 , 5

Continuous variable

For datasets grouped by class, we'll simply determine the *modal class*, which is done as follows:

1. We adjust the frequencies
2. The class with the greatest adjusted frequency is identified. It's called the "modal class" and may be non-unique.

Example. In our example with the farms, after adjusting the frequencies, we obtain the following table:

Area in ha	Number of farms
$]0;5]$	24
$]5;10]$	24
$]10;15]$	62
$]15;20]$	107
$]20;25]$	133
$]25;30]$	84
$]30;35]$	33
$]35;40]$	33

The modal class is therefore the fifth one. Hence, $M_o \in]20;25]$.

Exercise 10.21. Determine the modal class of the following dataset.

Classes	Frequency
$[0;2[$	3
$[2;4[$	8
$[4;6[$	15
$[6;8[$	14
$[8;10[$	6
$[10;12[$	2

Exercise 10.22. The table below shows the diameter of some items in mm.

Classes	Frequency
$]20;25]$	9
$]25;30]$	27
$]30;35]$	36
$]35;45]$	45
$]45;55]$	18
$]55;60]$	9
$]60;65]$	3
$]65;70]$	3

Determine the modal class.

Exercise 10.23. Determine the modal class of each of the following populations.

Ages	Frequency
$]0;20]$	72
$]20;65]$	180
$]65;100]$	43

Salary (in francs)	Number of employees
]0;80]	32
]80;100]	48
]100;260]	20

Exercise 10.24. Compute the modal classes of this distribution.

Classes	Frequency
$[18;20[$	107
$[20;22[$	110
$[22;24[$	91
$[24;28[$	220

10.5.3 Median

Example. In 2016, a Swiss man reads in the press that the FSO estimates the average monthly gross salary at 7'491 francs. He compares this to his salary of 6'942 francs and protests against the stinginess of his employer, from whom he claims an increase of income. But is the average salary a relevant indicator in this case? Certainly not. It is based on a large number of people earning little and a small number of managers earning indecent salaries rising to several million, resulting in an upward distortion of the average wage. Instead, our individual should ask himself whether he earns more or less than most of his compatriots. To answer that question, we'll consider the median. This indicator divides the population into two equal parts. Since the median gross salary in Switzerland is 6'502 francs in 2016 according to the FSO, he is rather favoured since he is part of the half of the population that earns the most!

Discrete variable

Definition. The *median*, denoted by M_e, is the value of the variable that halves the total number of values. It is the value of the character that corresponds to a cumulative frequency equal to 50%. In a population, there are as many individuals with a value below the median as there are individuals with a value above the median.

The median class of a continuous variable is the first class where the cumulative frequency reaches or exceeds 50%.

Example.

1. The numbers 3, 4, 4, 5, 6, 8, 8, 8, 10 have a median of $M_e = 6$.
2. With the numbers 5, 5, 7, 9, 11, 12, 15, 18, we notice that we have an even number of observations, so the median is $M_e = \dfrac{9+11}{2} = 10$.

Remark. We can see that the median corresponds to the value of the individual occupying the rank

$$m = \frac{N+1}{2}.$$

.

— If N is odd, this is a genuine individual occupying the entire m rank..

— If N is even, it is a virtual individual placed between ranks $N/2$ and $N/2+1$.

Example. Let us take again our example of the number of persons per household. To determine the value of the median, the increasing cumulative frequency must be calculated.

Outcomes	Frequency	Cumulative Frequency
1	5	5
2	9	14
3	15	29
4	10	39
5	6	45
6	3	48
8	2	50

The median is between rank 25 and 26. So, $M_e = 3$.

Exercise 10.25. The values below indicate the number of languages spoken by each individual based on a sample of 30 employees of an insurance company.

3	5	2	4	3	4	5	4	3	4
2	1	2	1	3	4	1	2	5	4
1	1	1	1	1	2	4	4	2	2

Compute the median.

Exercise 10.26. 20 packages weight sent in one hour were registered at a post office counter (rounded up to the nearest kilogram).

3	3	2	3	1	2	1	2	4	4
4	1	1	1	3	1	4	2	4	1

a) Group this dataset in a table.

b) Determine the average weight.

c) Determine the mode.

d) Determine the median weight.

Exercise 10.27. Compute the arithmetic mean, the median and the mode.

Outcomes	Frequency
10	2
11	3
12	7
13	9
14	14
15	8
16	3
17	1

Exercise 10.28. A company is composed of 7 employees and its boss. The table shows the monthly salaries in francs of these eight people.

Jean B.	12’750
Nicole C.	4’950
Alphonse G.	4’200
Jennifer Z.	8’720
Christian A.	4’800
Adrien A.	5’080
Kim D.	4’500
Natacha T.	4’750

a) Compute the mean.

b) Compute the median.

c) Another employee is hired.

   (a) What’s his/her salary if the arithmetic mean is now 6’100 francs? What’s the median now?

   (b) What could be his/her salary if the median is 4’800 francs?

Exercise 10.29. Give a statistical series whose number of observations is 3, the median 8, the arithmetic mean 7 and one of the values is 4.

Exercise 10.30. In a lottery, 9 grandmothers were asked how many grandchildren they had. The answers were: 10, 5, 6, 8, 9, 10, 6 and 12. Unfortunately, the last one’s answer was lost.

a) What did she answer if the average number of grandchildren is 8?

b) What could she have answered if the median is 8?

Continuous variable

In this course, we will simply determine the class to which the median belong.

Example. Let’s take back the example with the farms.

Classes	Frequency	Cumulative frequency
$]0;10]$	48	48
$]10;15]$	62	110
$]15;20]$	107	217
$]20;25]$	133	350
$]25;30]$	84	434
$]30;40]$	66	500

The median area is between those of the 250^{th} and 251^{st} individuals. These two farms belong to the class $]20;25]$.

Hence, $M_e \in]20;25]$.

Exercise 10.31. In which class is the median?

Classes	Frequency
$[0; 2[$	3
$[2; 4[$	8
$[4; 6[$	15
$[6; 8[$	14
$[8; 10[$	6
$[10; 12[$	2

Exercise 10.32. A company is running a big bowling tournament. Here is the scoreboard:

Classes	Frequency
$[120; 140[$	1
$[140; 160[$	9
$[160; 180[$	22
$[180; 200[$	51
$[200; 220[$	12
$[220; 240[$	5
Total	100

Compute the arithmetic mean and determine to what class does the median belong.

10.5.4 Comparison between the measures of central tendency

We can now make some rough comparisons between the three measures of central tendency.

Arithmetic mean

1. It is perhaps the most familiar measure of central tendency.
2. It requires more computations, but is algebraically expressed in a simple way.
3. It takes into account all the data and is therefore influenced by the extreme data of the distribution. If a distribution is highly skewed, this becomes a disadvantage that justifies the use of the median instead of the mean.
4. It is little influenced by the choice of classes, but cannot be calculated if there is an open class (e.g. a class of the type "80 and over").
5. It is easily algebraically manipulated because of its simple mathematical expression.
6. Its value is stable, i.e. it varies little from one sample to another, since it takes into account all the data. This is its highest quality for making statistical inference.
7. This is the most widely used measure of central tendency.

Mode

1. There may be more than one in a dataset. In such cases, the presence of more than one mode may be an indication that the study population is composed of distinct subgroups. Depending on the desired survey, this could be an invitation to split the population.
2. It's easy to determine.
3. It does not take into account all the data. It is not influenced by the extreme data of the distribution.
4. It can be greatly influenced by the choice of classes.
5. It is only really meaningful if the corresponding frequency is significantly higher than the others.
6. Its value is not stable, i.e. it varies greatly from one sample to another chosen from the same population.
7. This is the least used measure of central tendency.

Median

1. It comes from a simple conception of the centre idea.
2. It is not very difficult to calculate, but it is more difficult to express algebraically than the arithmetic mean.
3. It does not take into account all the data, but only the position of the data. It is therefore not influenced by the extreme data of the distribution.
4. It can be influenced by the choice of classes.
5. It is mainly used when the distribution of frequency is highly skewed or when we have open classes.
6. Its value is less stable than the arithmetic mean. This is because this value depends on only a small number of data selected from a sample.
7. It is used more than the mode, but less than the arithmetic mean.

Exercise 10.33. A radar control measured the speed of 10 drivers on a road limited to 60 km/h.

70 60 80 60 60 60 90 200 70 60

Which measure of central tendency best describes this situation?

Exercise 10.34. Let's see two samples.

Sample A	2	9	3	8	3	9	3	2	10	3
Sample B	9	2	2	8	3	10	3	9	3	8

a) What is the arithmetic mean and median of the two samples?

b) Which of these two measures is most appropriate to describe the situation?

Exercise 10.35. In our class, we are 10 students. In 3 different exams I only got 8 out of 20 points, which is quite a pity. Using the measure of central tendency, how am I going to manage to present these results to my employer so that my results don't look so bad?

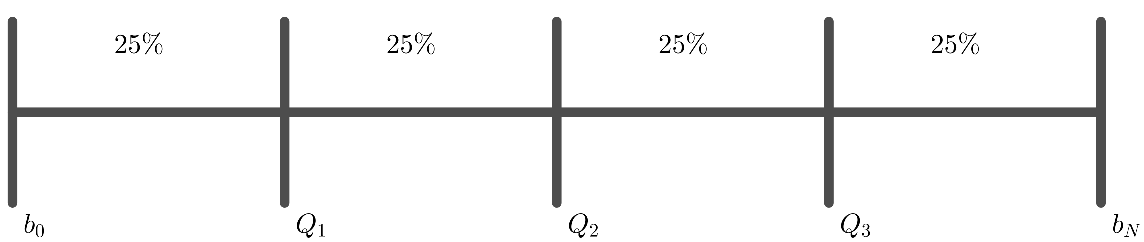
Examen	Me	Marc	Lili	Jojo	Bob	Fred	Karl	Léa	Luc	Jo
1st	8	2	2	2	9	9	9	9	10	19
2nd	8	2	3	4	5	7	9	9	18	19
3rd	8	2	7	7	7	10	11	12	18	19

10.6 Quartiles and box plot

10.6.1 Quartiles

Discrete variable

Definition. We call *quartiles* the character values that divide the total dataset into 4 groups of equal size. They are Q_1, Q_2 and Q_3. One quarter of the total staff therefore has a character less than Q_1. The second quartile $Q_2 = M_e$ is none other than the median. Finally, three-quarters of the population is below the value defined by the third quartile Q_3.



Remark. It should be kept in mind that there are many different methods for determining quartiles, which do not lead to the same results. In this course, we will calculate quartiles according to the method established by John Tukey in 1983.

We sort the N values in ascending order and cut it into two subsets where the median is calculated.

— If N is odd, there is a central value (the median), and the data are cut into two subsets by putting the median in each of the two sets. The quartile Q_1 is then the median of the first subset; the quartile Q_3 is the median of the second subset.

— If N is even, there are two central values (the median is the arithmetic mean of these two values), and we cut into two subsets by putting in each subset the corresponding central value. The quartile Q_1 is then the median of the first subset; the quartile Q_3 is the median of the second subset.

Example.

N	Values	Subsets	Q_1	Q_3	Rank of Q_1	Rank of Q_3
4	1 3 4 5	$\{1;3\}$ and $\{4;5\}$	2	4, 5	1, 5	3, 5
5	1 3 5 5 7	$\{1;3;5\}$ and $\{5;5;7\}$	3	5	2	4
6	1 3 4 6 7 9	$\{1;3;4\}$ and $\{6;7;9\}$	3	7	2	5
7	1 3 5 6 7 9 15	$\{1;3;5;6\}$ and $\{6;7;9;15\}$	4	8	2, 5	5, 5

Theorem.

— *If N is even, the rank of the quartile Q_1 is given by $\dfrac{N+2}{4}$ and the rank of Q_3 by $\dfrac{3N+2}{4}$.*

— *If N is odd, the rank of the quartile Q_1 is given by $\dfrac{N+3}{4}$ and the rank of Q_3 by $\dfrac{3N+1}{4}$.*

Example. Let's take our example with the number of persons per household.

Outcomes	Frequency	Cumulative frequency	Percent (in %)	Cumulative percent
1	5	5	10	10
2	9	14	18	28
3	15	29	30	58
4	10	39	20	78
5	6	45	12	90
6	3	48	6	96
8	2	50	4	100

We already know that $Q_2 = M_e = 3$. The quartile Q_1 is the value at the $\dfrac{50+2}{4} = 13$th position. So $Q_1 = 2$. For the third quartile Q_3, it's equal to the value at the $\dfrac{3\cdot 50+2}{4} = 38$th position. So $Q_3 = 4$.

Remark. Most of the time, when determining quartiles, it is not possible to find ranks that divide the population into four classes of equal size. In this case, the frequencies are converted to percents and the quartiles Q_1, M_e and Q_3 are defined by the values associated with the cumulative percent 25%, 50% and 75%.

Remark. It is possible to generalize the notion of quartile to *quantile of order n*. The other most commonly used quantiles are:

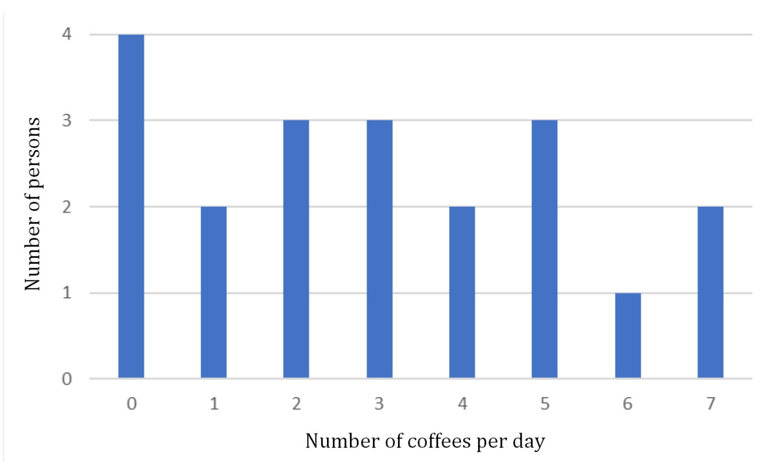
— *Deciles* D_1, D_2, ..., D_9 divide the total dataset into ten equal groups. One tenth of the population has a value less than D_1, and nine tenths have a value greater than D_1, ..., and so on. The decile D_5 is equal to the median;

— *Percentiles* C_1, C_2,. . . , C_{99} divide the total dataset into 100 equal groups.

Exercise 10.36. Compute the three quartiles of this distribution.

1	1	4	8	8	10	10	10	15	20
1	1	4	8	8	10	10	10	15	20
1	4	8	8	8	10	10	10	15	20
1	4	8	8	8	10	10	10	15	20
1	4	8	8	8	10	10	10	20	20

Exercise 10.37. Below is a histogram representing the daily coffee consumption of all the persons working in a company.



a) How many employees work in this company?

b) How many coffees does each employee drink on average?

c) Determine the median, the first and the third quartiles.

d) Determine the mode.

Exercise 10.38. Add a number to the following data set

$$-12; -10; -6; -3; 0; 3; 5; 8; 9; 11; 13$$

such that the median is 4 and the third quartile is 9.

Exercise 10.39. At a long jump competition, the performances below were measured in meters:

3,45	3,60	3,70	2,95	3,00	4,10	3,10	3,90
3,10	3,20	4,50	3,15	3,20	3,05	3,60	3,55
4,00	3,35	3,55	2,70	2,95	3,05	3,85	3,40

a) Determine the median, the maximum, the minimum, the quartiles.

b) Are the following statements correct? Justify.

   (a) Around 75% jumped further than $3,60$ m.
   (b) Around 50% jumped further than $3,375$ m.
   (c) Around 50% jumped less than $3,375$ m far.

Continuous variable

In this course, we will simply determine the classes in which the quartiles Q_1, Q_2 and Q_3 are contained.

Example. Let's compute the quartiles using the example with the farms.

Classes	Frequency	Cumulative frequency
$]0;10]$	48	48
$]10;15]$	62	110
$]15;20]$	107	217
$]20;25]$	133	350
$]25;30]$	84	434
$]30;40]$	66	500

We already know the second quartile $Q_2 = M_e \in]20;25]$.

The first quartile Q_1 corresponds to the area given by the observation at rank $\dfrac{500+2}{4} = 125,5$. Since it's in the third class, we have $Q_1 \in]15;20]$.

The third quartile Q_3 is defined by the area given by the observation at rank $\dfrac{3 \cdot 500+2}{4} = 375,5$. Since it's in the fifth class, we have $Q_3 \in]25;30]$.

Exercise 10.40. This table summarizes the height of students in a school (in cm).

Height of the student	Number of students
$[120;130[$	6
$[130;140[$	21
$[140;150[$	45
$[150;160[$	55
$[160;170[$	26
$[170;180[$	7

Determine the class to witch the quartiles Q_1 and Q_3 belong.

Exercise 10.41. A sample of 200 athletes is the database for the table below.

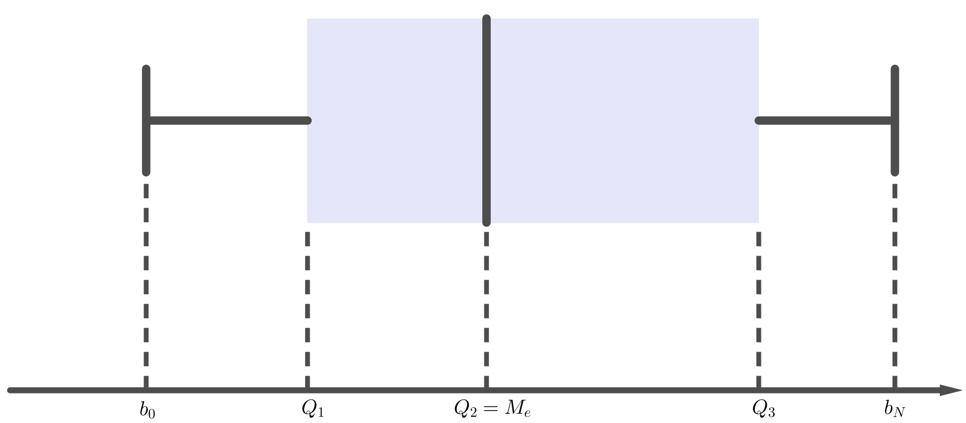
Height (in cm)	Frequency	Cumulative percent
$[150; 160[$		$0,2$
$[160; 170[$	60	
$[170; 180[$		$0,6$
$[180; 190[$	30	
$[190; 200[$		

a) Fill in the table.

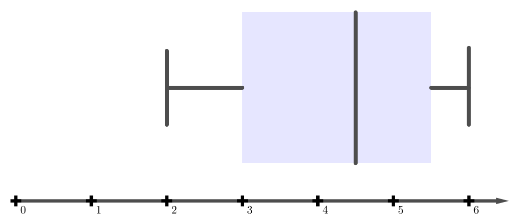
b) Determine the class that contains Q_1 and same for Q_3.

10.6.2 Box plot

Definition. The *Tukey diagram*, more commonly called *box plot*, is a representation of the quartiles Q_1, M_e, Q_3 and the extreme values b_0 and b_N of the distribution which gives a graphical information about the symmetry of the distribution.



Example. The following box plot represents the grades obtained in a test:



We can deduce the following information:

— The worst grade is 2 and the best one is 6.

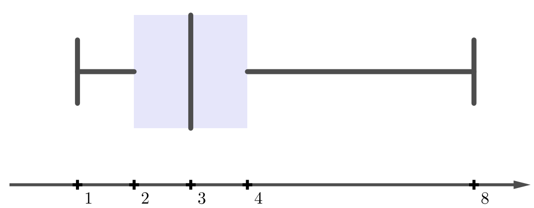
— 25% of the students got a grade less or equal to 3.

— Half of the students had at least $4,5$ (and also less or equal to $4,5$).

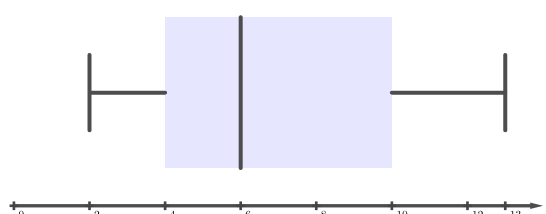
— 75% of the students had a grade less or equal to $5,5$.

— 50% of the students are located in a range of $2,5$.

Example. In the example with the number of persons per household, the box plot is



Exercise 10.42. In one company, a number of employees are questioned to find out how many e-mails they have sent in one day. The results are summarized by the box plot below.



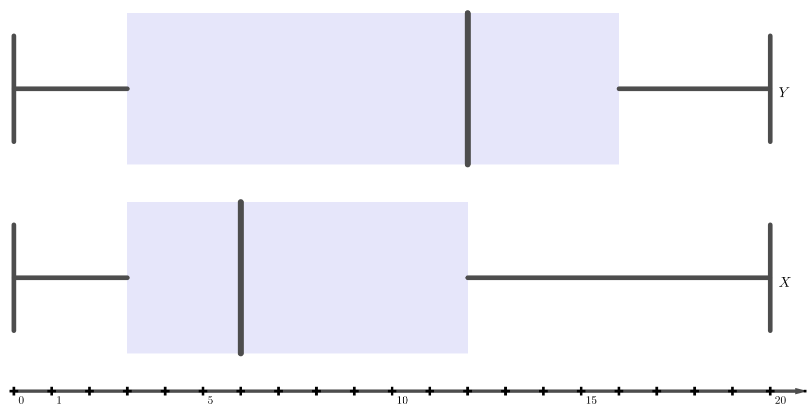
a) Determine the values of the three quartiles.

b) What's the greatest and the lowest number of e-mails sent?

c) Half of the employees surveyed sent more than how many e-mails?

d) What percentage of employees surveyed sent between 4 and 10 emails?

Exercise 10.43. In two cities X and Y, a sample of 1000 people were selected and asked how many cigarettes they smoked daily. The results are presented in the box plots below.



a) What is the value of the three quartiles Q_1, Q_2 and Q_3 for both cities?

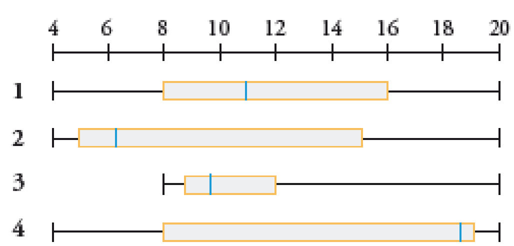
b) Which city smoke the most?

c) Can we say that only a quarter of the population of the city X smokes more than 3 cigarettes a day?

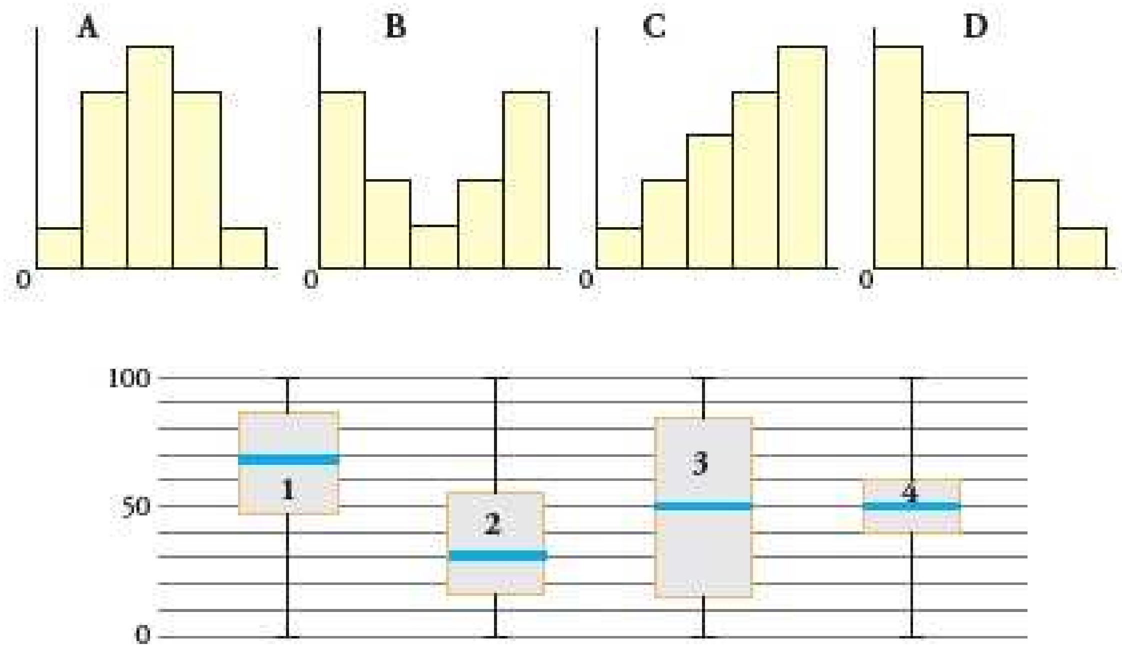
d) Can we say that half of the population of the city Y smokes less than 11 cigarettes a day?

Exercise 10.44. Match the following four distributions to the corresponding box plots below.

X	4	5	8	9	18	19	19	19	19	20
Y	8	9	9	9	9	10	11	12	18	20
Z	4	4	5	5	6	7	10	15	18	20
U	4	6	8	8	10	12	16	16	16	20



Exercise 10.45. Match each histogram to the corresponding box plots below.



Exercise 10.46. A weather station measured and recorded the outside temperature at noon every day throughout February. The result is the temperature table expressed in °C.

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Temperature	7	-2	0	1	-5	-9	-6	-5	-2	4	7	11	12	10

Day	15	16	17	18	19	20	21	22	23	24	25	26	27	28
Temperature	10	10	7	4	3	-2	-3	-6	-1	4	7	11	12	10

Draw the corresponding box plot.

Exercise 10.47. 3 classes of 34 students obtained the following points in the most recent test evaluated out of 20 points.

Class 1:

5	10	15	12	13	14	12
5	8	9	6	12	14	18
16	15	19	14	12	10	11
12	16	12	15	4	12	8
9	14	5	3	18	17	

Class 2:

8	8	9	10	8	9	10
11	8	9	12	8	12	11
10	12	11	10	12	10	11
12	9	8	7	13	10	11
12	11	10	8	10	11	

Class 3:

2	18	5	15	3	17	6
14	0	20	1	19	8	12
9	11	10	10	12	8	11
9	16	4	15	5	12	8
9	11	9	11	19	1	

a) Make the table with the frequencies for each class.

b) Draw the box plots.

c) What is the average score for each class?

d) On the light of the information obtained, indicate the class or classes that best fit the following criteria:

Criteria A: "The students have similar results. The class is homogeneous".
Criteria B: "The students have very different results. The class is heterogeneous".
Criteria C: "This class has the best results".
Criteria D: "At least 50% of the students have a score between 8 and 12".
Criteria E: "25% of the students at most get a score between 8 and 12".

10.7 Measures of spread

While measures of central tendency are generally necessary to describe a data set, they are not sufficient. Two different populations may have the same measures of central tendency and may differ significantly in the spread of individuals around these measures.

The two sets $A = \{6; 8; 10; 12; 14\}$ and $B = \{2; 6; 10; 14; 18\}$ have, for example, the same arithmetic mean and median of 10. However, the individuals that compose them are not distributed in the same way around this value. The set B is less regular or more spread than the set A. It is said that A and B do not have the same dispersion.

To compare two populations, measures of their dispersion are considered in addition to their central tendency. Typical measures of spread are: *the range, the variance and the standard deviation.*

10.7.1 Range

It's the easiest measure of spread.

Definition. *The range* is the difference between the greatest and the lowest value.

Example. In the farms example, the range is $R = 40 - 0 = 40$ ha.

Remark. Easy to calculate, this measure of dispersion is not very reliable since it takes into account only two marginal observations and neglects all the others.

Exercise 10.48. The following series of observations is a record of temperatures during the month of February in one city.

−10	−9	−5	−12	−8	−5	−2
5	2	3	5	6	2	0
1	−1	−2	0	−6	−10	−11
−9	−6	−4	−3	0	0	1

Compute the range.

10.7.2 Interquartile range

Definition. *The interquartile range* I_Q is defined by the difference of the extreme quartiles. In other words, we have

$$\boxed{I_Q = Q_3 - Q_1.}$$

Remark. This measure is more reliable than the range since it excludes the 50% of the lower and upper marginal values.

Definition. *The semi-interquartile range* Q is defined as the arithmetic mean of the differences between the quartiles and the median. In other words, we have

$$\boxed{Q = \frac{(Q_3 - M_e) + (M_e - Q_1)}{2} = \frac{Q_3 - Q_1}{2} = \frac{I_Q}{2}.}$$

Example. In our example of the number of persons per household, we have

$$I_Q = 4 - 2 = 2$$

and

$$Q = \frac{4 - 2}{2} = 1.$$

Remark. It's also possible to define *the interdecile range* I_D by

$$\boxed{I_D = D_9 - D_1.}$$

This defines an interval containing 80% of the population.

Exercise 10.49. The table below gives the capacity of the stadiums of the teams of the 2019-2020 Premier League football season.

Club	Stadium	Club	Stadium
AFC Bournemouth	11'464	Everton FC	39'571
Watford FC	20'877	Chelsea FC	41'798
Burnley FC	21'401	Aston Villa	42'788
Crystal Palace	26'255	Newcastle United	52'405
Norwich City	27'244	Liverpool FC	54'074
Brighton & Hove Albion	30'750	Manchester City	55'097
Wolverhampton Wanderers	31'700	West Ham United	60'000
Southampton FC	32'505	Arsenal FC	60'704
Sheffield United	32'702	Tottenham Hotspur	62'062
Leicester City	36'262	Manchester United	75'635

Give the value of Q_1, Q_2 and Q_3 and deduce the interquartile range and the semi-interquartile range.

Exercise 10.50. One company reported employee absence statistics for the past month.

Number of days absent	Number of employees
0	36
1	42
2	20
3	11
4	3
5	2
12	1

a) Compute the range and the semi-interquartile range.

b) Compute the proportion of employees who missed more than two days of work.

10.7.3 Variance and standard deviation

Discrete variable

Example. Two classes of 20 students took written mathematics tests, the results are presented in the tables below.

Grade x_i	Number of students n_i
1	0
$1,5$	0
2	0
$2,5$	0
3	0
$3,5$	3
4	7
$4,5$	8
5	1
$5,5$	1
6	0

Table 10.1: First class grades.

Grade y_i	Number of students n_i
1	0
$1,5$	1
2	0
$2,5$	2
3	4
$3,5$	0
4	0
$4,5$	3
5	6
$5,5$	2
6	2

Table 10.2: Second class grades.

In light of the above results, it's natural to wonder:

Which of these two classes performed better on this written test?

One way to answer that question is to calculate the arithmetic mean of each of the two classes:

$$\overline{x} = \frac{3,5 \cdot 3 + 4 \cdot 7 + 4,5 \cdot 8 + 5 \cdot 1 + 5,5 \cdot 1}{20} = 4,25$$

$$\overline{y} = \frac{1,5 \cdot 1 + 2,5 \cdot 2 + 3 \cdot 4 + 4,5 \cdot 3 + 5 \cdot 6 + 5,5 \cdot 2 + 6 \cdot 2}{20} = 4,25$$

These two arithmetic means $\overline{x}$ and $\overline{y}$ are equal, but the results are very different!

The arithmetic mean doesn't give any information on *the dispersion of the results* around the mean. To estimate it, we try to quantify how the scores are distributed around the mean.

We obtain:

$x_i - \overline{x}$	n_i
$-3,25$	0
$-2,75$	0
$-2,25$	0
$-1,75$	0
$-1,25$	0
$-0,75$	3
$-0,25$	7
$0,25$	8
$0,75$	1
$1,25$	1
$1,75$	0

$y_i - \overline{y}$	n_i
$-3,25$	0
$-2,75$	1
$-2,25$	0
$-1,75$	2
$-1,25$	4
$-0,75$	0
$-0,25$	0
$0,25$	3
$0,75$	6
$1,25$	2
$1,75$	2

The computation of the arithmetic mean of these differences is zero, because the negative differences are exactly balanced by the positive differences, which leads to zero information on dispersion. We then choose to calculate the square of the deviations from the mean.

The following distributions are then obtained:

$(x_i - \overline{x})^2$	n_i
10,5625	0
7,5625	0
5,0625	0
3,0625	0
1,5625	0
0,5625	3
0,0625	7
0,0625	8
0,5625	1
1,5625	1
3,0625	0

$(y_i - \overline{y})^2$	n_i
10,5625	0
7,5625	1
5,0625	0
3,0625	2
1,5625	4
0,5625	0
0,0625	0
0,0625	3
0,5625	6
1,5625	2
3,0625	2

Let's compute the arithmetic mean of $(\overline{x} - x_i)^2$ and $(\overline{y} - y_i)^2$:

$$\begin{aligned}
\overline{(x_i - \overline{x})^2} &= \frac{0,5625 \cdot 3 + 0,0625 \cdot 7 + 0,0625 \cdot 8 + 0,5625 \cdot 1 + 1,5625 \cdot 1}{20} \\
&= 0{,}2375
\end{aligned}$$

$$\begin{aligned}
\overline{(y_i - \overline{y})^2} &= \frac{7,5625 \cdot 1 + 3,0625 \cdot 2 + 1,5625 \cdot 4 + 0,0625 \cdot 3 + 0,5625 \cdot 6 + 1,5625 \cdot 2 + 3,0625 \cdot 2}{20} \\
&= 1{,}6375.
\end{aligned}$$

These numbers found are a *measure of the dispersion* of the grades around the arithmetic mean. We can therefore see that the grades of the first class are closer to the mean than those of the second class.

Definition. We call *variance* V of a statistical series the mean of the squares of the differences between all the data and their arithmetic mean. We therefore have

$$\boxed{V = \overline{(x_i - \overline{x})^2} = \frac{(x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + ... + (x_n - \overline{x})^2}{N}.}$$

Definition. We call *standard deviation* σ, the square root of the variance. In other words, we have

$$\boxed{\sigma = \sqrt{V}.}$$

Remark. The standard deviation is a more significant measure of dispersion than the variance. This is because if the x_i data represent a distance expressed in meters, V is in m^2 while the standard deviation is expressed in meters.

Example. Let's see with the numbers -4, 3, 9, 11 and 17.

Their arithmetic mean is

$$\overline{x} = \frac{-4+3+9+11+17}{5} = 7,2.$$

From the table

	x_i	$x_i - \overline{x}$	$(x_i - \overline{x})^2$
	-4	$-11,2$	$125,44$
	3	$-4,2$	$17,64$
	9	$1,8$	$3,24$
	11	$3,8$	$14,44$
	17	$9,8$	$96,04$
Total	36	0	$256,8$

we can deduce the value of the variance

$$V = \frac{256,8}{5} = 51,36$$

and the standard deviation

$$\sigma = \sqrt{51,36} \cong 7,167.$$

Example. Let's calculate the variance and standard deviation of our example with the number of persons per household. For this purpose, the table below should be established. Note that the arithmetic mean was 3,44.

Outcomes x_i	Frequency n_i	Deviations $x_i - \overline{x}$	Square of the deviations $(x_i - \overline{x})^2$	Product $n_i \cdot (x_i - \overline{x})^2$
1	5	$-2,44$	$5,9536$	$29,768$
2	9	$-1,44$	$2,0736$	$18,6624$
3	15	$-0,44$	$0,1936$	$2,904$
4	10	$0,56$	$0,3136$	$3,136$
5	6	$1,56$	$2,4336$	$14,6016$
6	3	$2,56$	$6,5536$	$19,6608$
8	2	$4,56$	$20,7936$	$41,5872$
Total	50			130,32

we deduce the variance

$$V = \frac{130,32}{50} = 2,6064$$

and therefore

$$\sigma = \sqrt{V} \cong 1,614.$$

Exercise 10.51. Compute the variance and the standard deviation of the following numbers: -9, -4, 1, 7, 10, 21.

Exercise 10.52. A casino asked its casino dealer to record for 60 consecutive days the number of times a day that 0 comes out in the roulette game. He obtained the following data.

Number of 0 per day	Number of days
7	1
8	3
9	6
10	9
11	14
12	11
13	7
14	6
15	2
16	1

a) Compute the measures of central tendency.

b) Compute the variance and the standard deviation.

Continuous variable

Similar to calculating the arithmetic mean, all individuals of a class $]b_{i-1}; b_i]$ are attributed the mid value $c = \dfrac{b_{i-1} + b_i}{2}$.

Example. In our farm example, this deviation is calculated using the following table. The arithmetic mean was 21.

Classes x_i	Mid values c_i	Frequency n_i	Square of deviations $(c_i - \overline{x})^2$	Products $n_i \cdot (c_i - \overline{x})^2$
$]0; 10]$	5	48	256	12288
$]10; 15]$	12, 5	62	72, 25	4479, 5
$]15; 20]$	17, 5	107	12, 25	1310, 75
$]20; 25]$	22, 5	133	2, 25	299, 25
$]25; 30]$	27, 5	84	42, 25	3549
$]30; 40]$	35	66	196	12936
Total		500	196	34862, 5

The variance is therefore $V = \dfrac{34862,5}{500} = 69,725 \text{ ha}^2$ and the standard deviation is given by $\sigma = \sqrt{69,725 \text{ ha}^2} \cong 8,35$ ha.

Exercise 10.53. Determine the variance and the standard deviation of the following populations.

Ages	Frequency
$]0;20]$	72
$]20;65]$	180
$]65;100]$	43

Salary (in francs)	Number of employees
]0;80]	32
]80;100]	48
]100;260]	20

Alternative calculation method

Calculating the variance (and therefore the standard deviation) is not always convenient. Especially when the mean is a number which is only approximated with a limited decimal development. However, the calculations can be simplified as follows.

Theorem. *The variance V can be obtained by calculating the difference between the mean $\overline{x^2}$ of the squares of the x_i and the square of their mean $\overline{x}^2$. Therefore, we have*

$$\boxed{V = \overline{x^2} - \overline{x}^2.}$$

Proof. We have

$$\begin{aligned}
V &= \frac{(x_1-\overline{x})^2 + (x_2-\overline{x})^2 + \ldots + (x_N-\overline{x})^2}{N} \\
&= \frac{(x_1^2 - 2x_1\overline{x} + \overline{x}^2) + (x_2^2 - 2x_2\overline{x} + \overline{x}^2) + \ldots + (x_N^2 - 2x_N\overline{x} + \overline{x}^2)}{N} \\
&= \frac{x_1^2 + x_2^2 + \ldots + x_N^2}{N} - \frac{2\overline{x}(x_1 + x_2 + \ldots + x_N)}{N} + \frac{\overline{x}^2 + \overline{x}^2 + \ldots + \overline{x}^2}{N} \\
&= \overline{x^2} - 2\overline{x}\cdot\overline{x} + \overline{x}^2 \\
&= \overline{x^2} - 2\overline{x}^2 + \overline{x}^2 \\
&= \overline{x^2} - \overline{x}^2.
\end{aligned}$$

□

Remark. This second formulation will be preferred to the first whenever the terms $x_i - \overline{x}$ are more complicated than the terms x_i. This case frequently occurs when the average is not an integer.

Example. Let's take again the example of the numbers -4, 3, 9, 11 and 17 of arithmetic mean $7,2$.

From this table,

	x_i	x_i^2
	-4	16
	3	9
	9	81
	11	121
	17	289
Total	36	516

we deduce the variance $V = \dfrac{516}{5} - 7,2^2 = 51,36$ and the standard deviation $\sigma = \sqrt{51,36} \cong 7,167$. The results are the same as above.

Example. Let's take the farms example.

Classes x_i	Mid values c_i	Frequency n_i	Square of mid values c_i^2	Products $n_i \cdot c_i^2$
$]0;10]$	5	48	25	1200
$]10;15]$	$12,5$	62	$156,25$	$9687,5$
$]15;20]$	$17,5$	107	$306,25$	$32768,75$
$]20;25]$	$22,5$	133	$506,25$	$67331,25$
$]25;30]$	$27,5$	84	$756,25$	63525
$]30;40]$	35	66	1225	80850
Total		500		$255362,5$

We deduce that $\overline{x^2} = \dfrac{255362,5}{500} = 510,725$. As $\overline{x} = 21$, $\overline{x}^2 = 441$, we have

$$V = \overline{x^2} - \overline{x}^2 = 510,725 - 441 = 69,725 \text{ ha}^2$$

and the standard deviation is therefore $\sigma = \sqrt{69,725 \text{ ha}^2} \cong 8,35$ ha.

Exercise 10.54. Use this method for the exercises 10.51 and 10.53.

Exercise 10.55. Let's consider the following data:

$$2 \quad 3 \quad 5 \quad 5 \quad 4 \quad 4 \quad 5 \quad 2 \quad 2 \quad 4$$

Compute x_{min}, x_{max}, Q_1, M_e, Q_3, $\overline{x}$, $\overline{x^2}$, V and σ.

Exercise 10.56. Automobile speed recordings gave the following results.

Speed	Number
$]30;35]$	15
$]35;40]$	39
$]40;45]$	65
$]45;50]$	87
$]50;55]$	70
$]55;60]$	22
$]60;65]$	12

Compute the arithmetic mean and the standard deviation.

Exercise 10.57. A survey on annual salaries of employees in a large company revealed the following results.

Classes	Frequency
$[20'000;22'000[$	80
$[22'000;24'000[$	130
$[24'000;26'000[$	340
$[26'000;28'000[$	210
$[28'000;30'000[$	120
$[30'000;32'000[$	40

a) Compute the arithmetic mean.

b) Compute the variance and the standard deviation.

10.7.4 Coefficient of variation

To describe a distribution, a measure of central tendency and a measure of spread are generally used. For example, one can give its median and its semi-interquartile range. However, in the vast majority of cases, a distribution is described by its mean and standard deviation. The mean indicates around which value the data are located, while the standard deviation gives an idea of the dispersion. This idea of dispersion must, however, be embedded in the data. If the standard deviation of a distribution is equal to 10, can we say that the distribution is widely dispersed? Of course, this depends on the order of size of the data. For example, if the data processed are of the order of 2000, this standard deviation is really small and the data are surely very concentrated. On the other hand, if the data are of the order of 12, for example, the standard deviation is large and the data are relatively dispersed. It is therefore useful to measure the *relative dispersion*.

Definition. The *coefficient of variation* C of a variable is the ratio of the standard deviation to the arithmetic mean expressed in percentage:

$$\boxed{C = \frac{\sigma}{\overline{x}}}$$

Remark. If one wishes to make a statement on the dispersion of a series, the following description is generally accepted:

Coefficient of variation	Dispersion
0 to 10%	Low
10 to 20%	Average
More than 20%	High

Example. In our farms example, this coefficient is

$$C = \frac{\sigma}{\overline{x}} \cong \frac{8,35}{21} \cong 0,398 = 39,8\%.$$

Hence, the values are widely spread.

Exercise 10.58. During the month of February, on the U.S. electronics stock market, the average daily closing price was 1'500$ with a standard deviation of 50$ for the class A of stocks, while for the class B, during the same period, the average was 500$ with a standard deviation of 30$.

a) Determine the coefficient of variation for A.

b) Same for B.

c) Compare the results and interpret the result.

Exercise 10.59. The following numbers indicate the mass in grams of 20 chicks:

$$67; 73; 76; 82; 60; 62; 60; 62; 55; 64; 64; 55; 75; 66; 61; 69; 72; 73; 54; 59.$$

Compute the arithmetic mean, the median, the standard deviation and the coefficient of variation.

Exercise 10.60. Let's consider this population.

Amount spent in francs	Number of costumers
$[0; 20[$	141
$[20; 40[$	239
$[40; 60[$	451
$[60; 80[$	578
$[80; 100[$	321
$[100; 200[$	294

Compute the standard deviation and the coefficient of variation.

10.7.5 Comparison between the measures of spread

Range

1. It is very simple to calculate and interpret.
2. It does not take into account all the data and implies only the extreme values.
3. It is used to give a rough and quick idea of dispersion and to determine class size when grouping into classes.
4. Its value is not stable, i.e. it varies greatly from one sample to another selected from the same population.
5. It's very little used.

Semi-interquartile range

1. It is very simple to calculate and interpret.
2. It does not take into account all data and is therefore not influenced by extreme values.
3. It is used when the distribution of the frequency is highly skewed. In this case, the median is used as a measure of central tendency.
4. Its value is less stable than the variance or standard deviation.
5. It is rarely used in general.

Interquartile range

1. It has the same properties as the semi-interquartile range.

Standard deviation

1. Its calculation takes longer and its interpretation is less obvious.
2. It takes into account all the data.
3. It can be manipulated algebraically quite well. It is therefore found in many inferential statistics calculations.
4. Its value is stable from sample to sample.
5. It is, together with variance, the most widely used measure of dispersion.

Variance

1. The variance has the same properties as the standard deviation.
2. The presence of squares gives more weight to large deviations. It is thus strongly influenced by extreme data.

Remark. The choice of the measure of central tendency implies the choice of the measure of dispersion:

$$\begin{array}{lcl} \text{mode} & \leftrightarrow & \text{range} \\ \text{median} & \leftrightarrow & \text{Semi-interquartile range} \\ \text{mean} & \leftrightarrow & \text{standard deviation} \end{array}$$

10.8 Solutions

Exercise 10.1.

a) The population studied is "new passenger cars registered in 2015".

b) The variable is the "type of energy consumed by those cars".

Exercise 10.2.

a) (Quantitative) continuous.

b) (Quantitative) discrete.

c) (Qualitative) nominal.

d) (Quantitative) continuous.

e) (Qualitative) ordinal.

f) (Quantitative) discrete.

g) (Qualitative) ordinal.

h) (Qualitative) nominal.

i) (Quantitative) continuous.

j) (Quantitative) discrete.

k) (Quantitative) discrete.

l) (Qualitative) ordinal.

m) (Qualitative) nominal.

n) (Quantitative) discrete.

o) (Quantitative) continuous.

p) (Qualitative) ordinal.

Exercise 10.3.

a) The population: the employees of a company. The variable: the political party they voted for.

b) The outcomes: $\{\mathrm{PS}; \mathrm{PLR}; \mathrm{PDC}; \mathrm{UDC}; \mathrm{verts}\}$.

c) It's an ordinal variable.

Exercise 10.4.

a) The population: the 80 students of the university. The variable: the number of points obtained.

b) The outcomes: $\{2; 3; ...; 10\}$.

c) It's a discrete variable.

Exercise 10.5.

Classes	Mid values	Frequency	Percent
$[30;45[$	37,5	1	$1,\overline{6}\%$
$[45;60[$	52,5	2	$3,\overline{3}\%$
$[60;75[$	67,5	9	15%
$[75;90[$	82,5	35	$58,\overline{3}\%$
$[90;105[$	97,5	12	20%
$[105;120[$	112,5	1	$1,\overline{6}\%$
Total		60	100%

Exercise 10.6.

a) False, it's the cultural spending.

b) True.

c) False, it's a nominal variable.

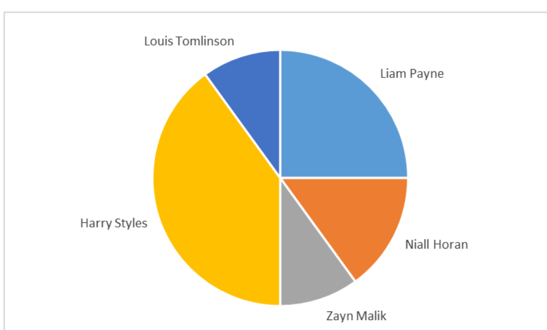
d) False. The total should be and is 100%.

Exercise 10.7.

a)

Singer	Frequency	Percent (in %)	Angle (in degrés)
Liam Payne	350	25	90°
Niall Horan	210	15	54°
Zayn Malik	140	10	36°
Harry Styles	560	40	144°
Louis Tomlinson	140	10	36°

b)

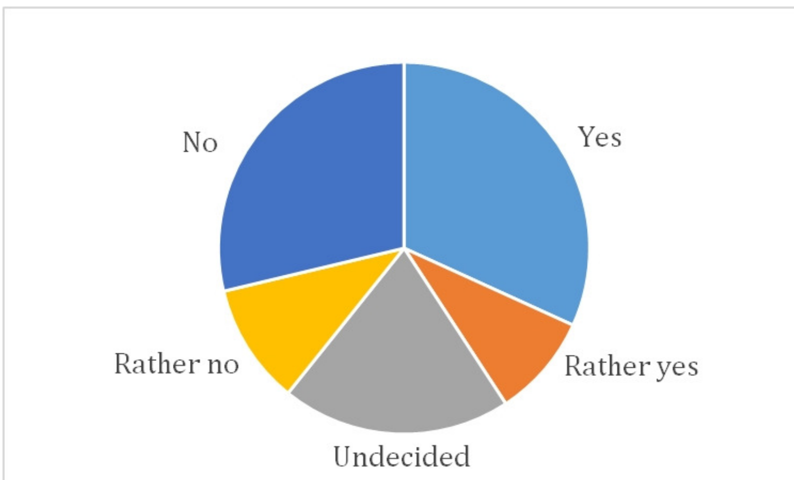


Exercise 10.8.

a)

Opinion	Frequency
Yes	$31,79\%$
Rather yes	$8,97\%$
Undecided	20%
Rather no	$10,51\%$
No	$28,71\%$

b)



Exercise 10.9.

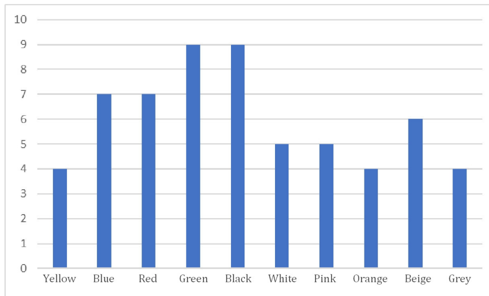
a) The population is "the dresses in the fashion store" and the statistical variable is "the colour of the dresses".

b) The outcomes are: $\{$yellow; blue; red; green; black; white; pink; orange; beige; grey$\}$.

c)

Colour	Frequency	Percent
Yellow	4	$6,7\%$
Blue	7	$11,6\%$
Red	7	$11,6\%$
Green	9	$15,0\%$
Black	9	$15,0\%$
White	5	$8,3\%$
Pink	5	$8,3\%$
Orange	4	$6,7\%$
Beige	6	$10,0\%$
Grey	4	$6,7\%$
Total	60	100%

d)



Exercise 10.10.

a) 30 employees of a small company.

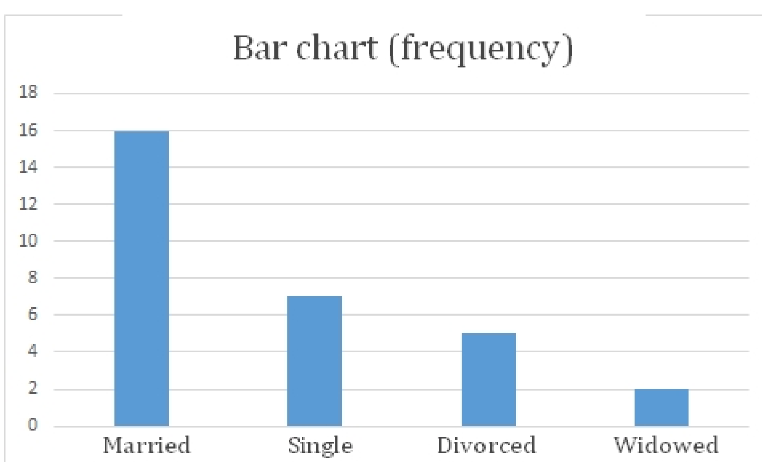
b) Their civil status.

c) {Married, Single, Divorced, Widowed}.

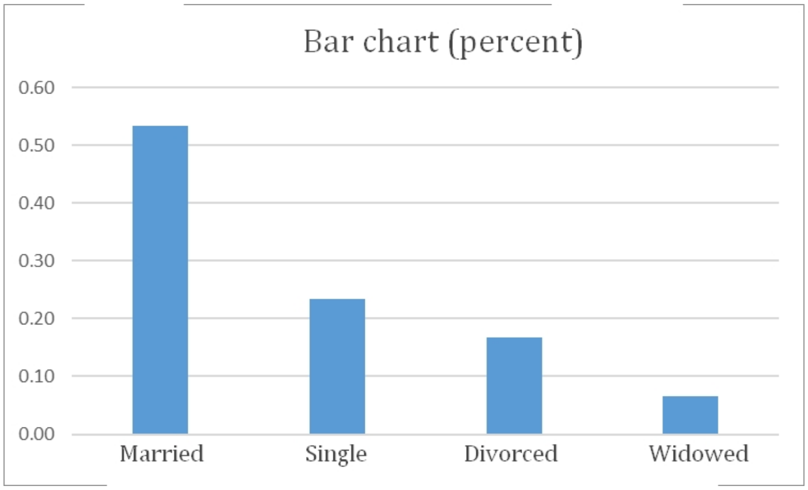
d)

Outcomes	Frequency	Percent
Married	16	0, 53
Single	7	0, 22
Divorced	5	0, 17
Widowed	2	0, 07
Total	30	1

e)



f)



g) Their are similar.

Exercise 10.11.

a) The population is "the farms in 2015" and the surface is the variable.

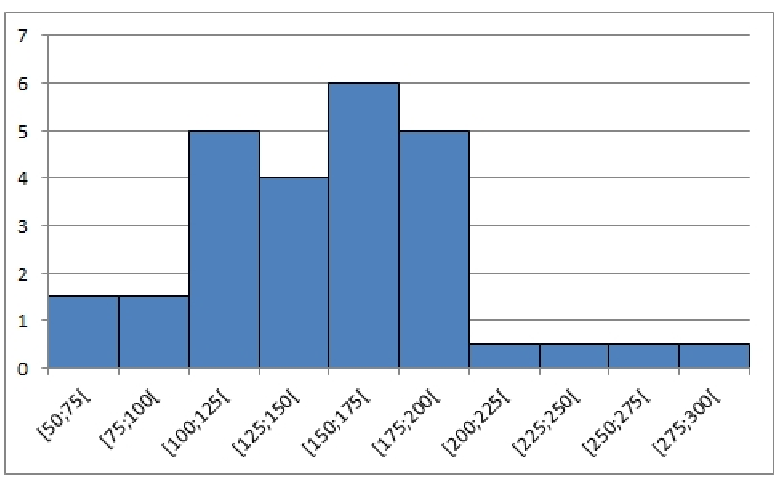
b) Classes.

c) Because there are many different outcomes. This makes it easier to read.

d) Continuous variable.

e) For example a histogram.

Exercise 10.12.



Exercise 10.13. This graph will give a biased picture of reality since the horizontal scale is not linear.

Exercise 10.14.

- The origin on the vertical axis is not at 0, but just above 5000, giving the impression of a multiplication by 6 of naturalizations between 1991 and 1993 (instead of a multiplication by 2.5).
- The year 1990 was chosen as a starting point because it represents the year with the lowest number of naturalizations since then.
- Finally, it should be noted that the values are absolute, and do not take into account the growth of the population from one year to the next.

Exercise 10.15.

a) The proportion of foreigners in Switzerland in 2010 was

$$\frac{2'092'309}{7'700'000 + 200'000 + 212'566 + 40'797} \cong 25,7\%.$$

b) To the 1'638'949 foreigners being part of 7,7 millions, we add

$$200'000 + 212'566 + 40'797 = 453'363,$$

that should also be added to the 7,7 million.

c) In addition to the 2'092'309 foreigners obtained above, the 550'000 naturalized between 1985 and 2010 were added, giving a total of 2'642'309. We then calculated the ratio

$$\frac{2'642'309}{7'700'000} \cong 34,3\%.$$

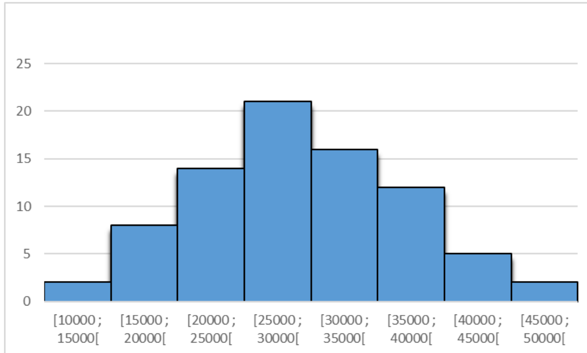
This calculation presents two issues. First, the total is incorrect, as in the previous point. On the other hand, this approach is based on the assumption that the 550'000 people who have been naturalized over the last 25 years are all alive and still living in Switzerland!

Exercise 10.16.

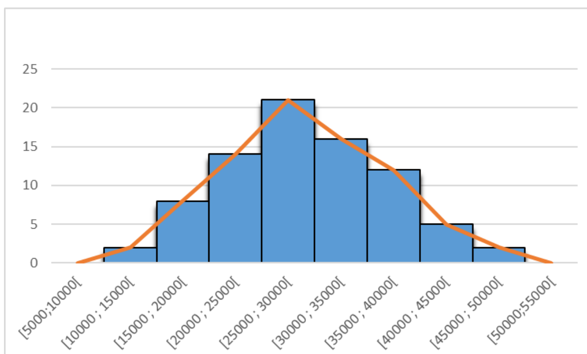
a)

Classes (Salary)	Mid values	Frequency	Percent
$[10000; 15000[$	12500	2	$2,5\%$
$[15000; 20000[$	17500	8	10%
$[20000; 25000[$	22500	14	$17,5\%$
$[25000; 30000[$	27500	21	$26,25\%$
$[30000; 35000[$	32500	16	20%
$[35000; 40000[$	37500	12	15%
$[40000; 45000[$	42500	5	$6,25\%$
$[45000; 50000[$	47500	2	$2,5\%$
Total		80	100

b)



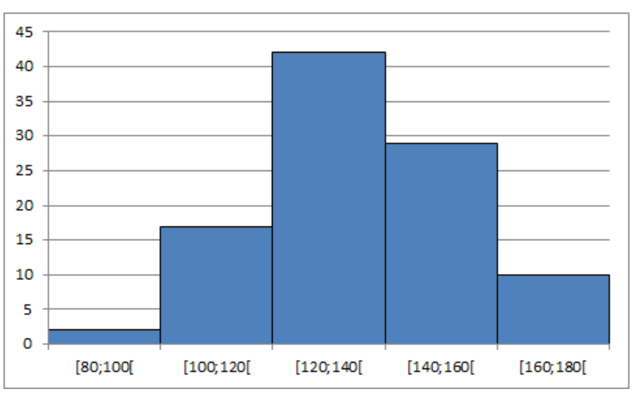
c)

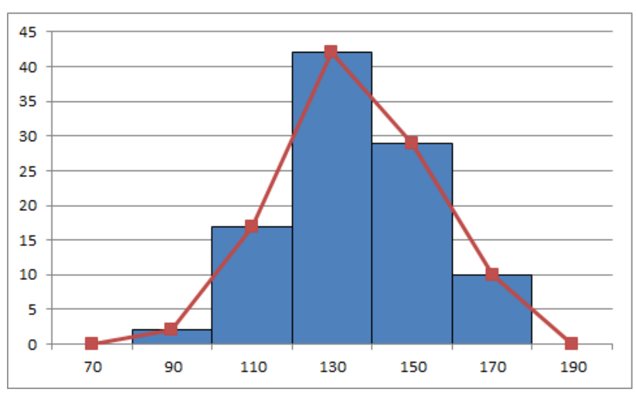


Exercise 10.17.

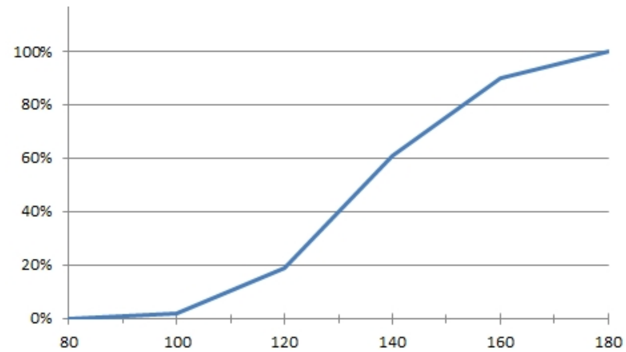
a) Continuous variable.

b)





c)



d) 162.

e) 141.

Exercise 10.18. $\overline{x} \cong 6,46$.

Exercise 10.19. $\overline{x} = 40,4$, $\overline{x} = 92$.

Exercise 10.20.

a) $M_o = 7$.

b) $M_o = 4$ and $M_o = 8$.

Exercise 10.21. $M_o \in [4;6[$.

Exercise 10.22. The modal class is $]30;35]$.

Exercise 10.23. $M_o \in]20;65]$, $M_o \in]80;100]$.

Exercise 10.24. The modal classes are $[20;22[$ and $[24;28[$.

Exercise 10.25. $M_e = 2,5$.

Exercise 10.26.

a)

Weight	Frequency	Cumulative Frequency
1	7	7
2	4	11
3	4	15
4	5	20

b) $\overline{x} = 2,35$ kg.

c) $M_o = 1$ kg.

d) $M_e = 2$ kg.

Exercise 10.27. $\overline{x} \cong 13,51$, $M_e = 14$ and $M_o = 14$.

Exercise 10.28.

a) 6'218,75 francs.
b) 4'875 francs.
c) (a) 5'150 francs and the median becomes 4'950 francs.
   (b) Any value between 0 and 4'800 francs.

Exercise 10.29. 4, 8 and 9.

Exercise 10.30.

a) 6.
b) She could have answered any number less than or equal to 8.

Exercise 10.31. $M_e \in [4; 6[$.

Exercise 10.32. $\overline{x} = 185,8$ and $M_e \in [180; 200[$.

Exercise 10.33. The median, possibly the mode, but definitely not the mean.

Exercise 10.34.

a) Sample A: $\overline{x} = 5,2$ and $M_e = 3$
   Sample B: $\overline{x} = 5,7$ and $M_e = 5,5$.
b) The arithmetic mean.

Exercise 10.35. On the first exam, I'm above the average. On the second, I did better than half of the students. On the third, I scored more than the most frequent number of points, which is 7.

Exercise 10.36. $Q_1 = 8$, $Q_2 = 9$ and $Q_3 = 10$.

Exercise 10.37.

a) 20.
b) 3.
c) $M_e = 3$, $Q_1 = 1$ and $Q_3 = 5$.
d) $M_o = 0$.

Exercise 10.38. 9.

Exercise 10.39.

a) $M_e = 3,375$ m, Maximum $= 4,5$ m, Minumum $= 2,7$ m, $Q_1 = 3,075$ m and $Q_3 = 3,65$ m.
b) (a) False.
   (b) True.
   (c) True.

Exercise 10.40. $Q_1 \in [140; 150[$, $Q_3 \in [150; 160[$.

Exercise 10.41. a)

Height (en cm)	Frequency	Cumulative percent
$[150;160[$	40	$0,2$
$[160;170[$	60	$0,5$
$[170;180[$	20	$0,6$
$[180;190[$	30	$0,75$
$[190;200[$	50	1

b) $M_o \in [160;170[$, $M_e = 170$, $Q_1 \in [160;170[$ and $Q_3 = 190$.

Exercise 10.42.

a) $Q_1 = 4$, $Q_2 = M_e = 6$ and $Q_3 = 10$.

b) Between 2 and 13 emails.

c) More than 6 emails.

d) 50%.

Exercise 10.43.

a) For X: $Q_1 = 3$, $Q_2 = 6$ and $Q_3 = 12$.
For Y: $Q_1 = 3$, $Q_2 = 12$ and $Q_3 = 16$.

b) The city Y.

c) No.

d) No.

Exercise 10.44.

$$\begin{array}{ccc} 1 & \leftrightarrow & U \\ 2 & \leftrightarrow & Z \\ 3 & \leftrightarrow & Y \\ 4 & \leftrightarrow & X \end{array}$$

Exercise 10.45.

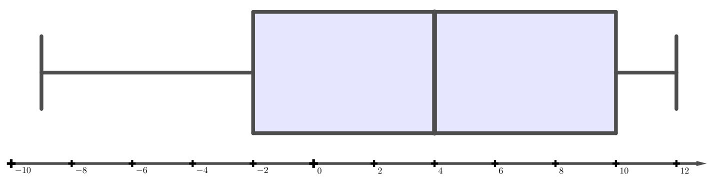
$A \leftrightarrow 4$.

$B \leftrightarrow 3$.

$C \leftrightarrow 1$.

$D \leftrightarrow 2$.

Exercise 10.46.

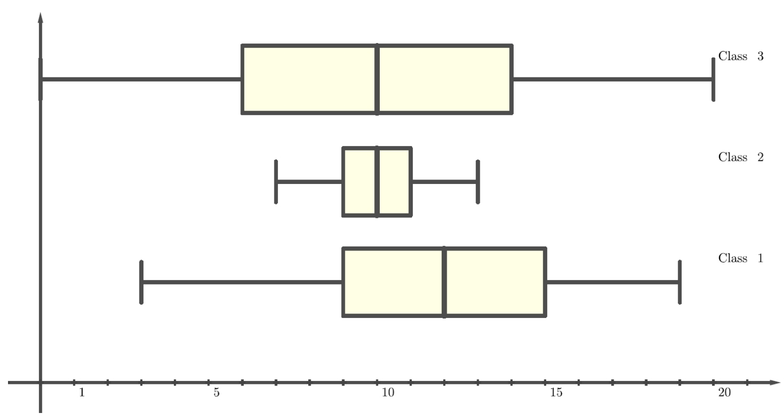


Exercise 10.47.

a)

Points	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Class 1	0	0	0	1	1	3	1	0	2	2	2	1	7	1	4	3	2	1	2	1	0
Class 2	0	0	0	0	0	0	0	1	7	4	8	7	6	1	0	0	0	0	0	0	0
Class 3	1	2	1	1	1	2	1	0	3	4	2	4	3	0	1	2	1	1	1	2	1

b)



c) $\overline{x_1} \cong 11,618$, $\overline{x_2} \cong 10,029$ and $\overline{x_3} = 10$.

d)

Criteria A: Class 2.
Criteria B: Class 3.
Criteria C: Class 1.
Criteria D: Class 2.
Criteria E: None.

Exercise 10.48. $R = 18$.

Exercise 10.49. $Q_1 = 28'997$, $Q_2 = 37'916,5$, $Q_3 = 54'585,5$, $I_Q = 25'588,5$ and $Q = 12'794,25$.

Exercise 10.50.

a) $R = 12$ and $Q = 1$.

b) $14,78\%$.

Exercise 10.51. $V = 95,\overline{8}$ and $\sigma \cong 9,79$.

Exercise 10.52.

a) $\overline{x} = 11,3\overline{6}$, $M_o = 11$ and $M_e = 11$.
b) $V = 3,63\overline{2}$ and $\sigma \cong 1,906$.

Exercise 10.53.

a) $V \cong 486,6$ and $\sigma \cong 22,059$.
b) $V = 2416$ and $\sigma \cong 49,153$.

Exercise 10.54.

Exercise 10.55. $b_0 = 2$, $b_{10} = 5$, $Q_1 = 2$, $M_e = 4$, $Q_3 = 5$, $\overline{x} = 3,6$, $\overline{x^2} = 14,4$, $V = 1,44$ and $\sigma = 1,2$.

Exercise 10.56. $\overline{x} = 46,887$ km/h and $\sigma \cong 7,056$ km/h.

Exercise 10.57.

a) $\overline{x} = 25'608,696$.
b) $V \cong 6'151'228,733$ and $\sigma \cong 2'480,167$.

Exercise 10.58.

a) $C_A = 3,\overline{3}\%$.
b) $C_B = 6\%$.
c) In both cases, the dispersion is low, although it is higher for the stock A, despite the fact that its standard deviation is lower than the one of B.

Exercise 10.59. $\overline{x} = 65,45$ g, $M_e = 64$ g, $\sigma \cong 7,57$ g and $C \cong 11,57\%$.

Exercise 10.60. $\sigma \cong 38,99$ and $C \cong 54,58\%$.

10.9 Chapter objectives

At the end of this chapter, the student should be able to

10.1 ☐ Master the vocabulary used in statistics.
10.2 ☐ Determine the type of the variable.
10.3 ☐ Group raw data into a table with frequency.
10.4 ☐ Compute the relative frequency of the outcomes.
10.5 ☐ Create a pie chart.
10.6 ☐ Create a bar chart.
10.7 ☐ Create a histogram.
10.8 ☐ Make a critical judgement on a diagram or graph.
10.9 ☐ Create a frequency polygon.
10.10 ☐ Compute the increasing cumulative frequency and the decreasing cumulative frequency.
10.11 ☐ Create a cumulative frequency polygon.
10.12 ☐ Compute the arithmetic mean (discrete and continuous variable).
10.13 ☐ Compute the mode (discrete and qualitative variable).
10.14 ☐ Determine the modal class (continuous variable).
10.15 ☐ Compute the median (discrete variable).
10.16 ☐ Be able to use the most appropriate measure of central tendency.
10.17 ☐ Determine the median class (continuous variable).
10.18 ☐ Compute the quartiles (discrete variable).
10.19 ☐ Give the classes to which the quartiles belong (continuous variable).
10.20 ☐ Create a box plot (discrete variable).
10.21 ☐ Interpret a box plot.
10.22 ☐ Compute the range (discrete and continuous variable).
10.23 ☐ Compute the interquartile range (discrete variable).
10.24 ☐ Compute the semi-interquartile range (discrete variable).
10.25 ☐ Compute the variance and the standard deviation (discrete and continuous variable).
10.26 ☐ Compute the coefficient of variation and interpret it (discrete and continuous variable).

Chapter 11

Revisions

11.1 Sets of numbers

Exercise 11.1. Compute without calculator.

a) $15 + 4 \cdot 6$
b) $45 : 5 - 81 : 9$
c) $17 + 35 - 72 : 9$
d) $120 : 2 \cdot 6 : 40$
e) $67 - 48 : 8 \cdot 6$
f) $7 \cdot 8 \cdot 2 - 64 : 8$
g) $240 \cdot 10 : 5 + 17 \cdot 0$
h) $132 : 12 + 5 \cdot 4 \cdot 6$
i) $230 - 180 : 9 + 1$
j) $72 : 8 + 4 \cdot 5 - 12$
k) $3^2 + 6^2 \cdot 5 - 4 : 4$
l) $3 \cdot 7 - 48 : 4 + 3 \cdot 10$
m) $4(13 - 24 : 4) - 12$
n) $6^2 - 3 \cdot 5 + 2 \cdot 5 \cdot (5 - 2)$
o) $15 + 8 \cdot 4 - [5 \cdot 3 + (2 + 3) \cdot 4 - 2]$
p) $4 + 5 \cdot 2 - [2 \cdot (5 \cdot 4 + 1 - 21) + 2]$

Exercise 11.2. Compute without calculator.

a) $(-5 + 5) : (3 - 2)$
b) $(34 - 45)(145 : (-5))$
c) $72 : (-8) - 15$
d) $12 - (-14 - 13) + (19 - 50)$
e) $(-24) : (-8) \cdot (-7) + (+3) \cdot (+11)$
f) $[-5 - (-3)] : [7 - (-7) \cdot (-1)]$
g) $(+3) \cdot [(-7) + (+11)] : [(-2) - (-8)]$
h) $[-11 + 5 \cdot 3 + 2 \cdot (-2)] : [-8 - 4 \cdot (-3) - 4]$
i) $-(50 - 30) + [(3 + 9 - 13)(-1 - 2) - (-20)]$
j) $(3 - 18 + 15) - \{2 - [-5 - (-4 - 14) + 11] - 3\}$

Exercise 11.3. Compute.

a) $-|-5|$
b) $|-3| + |-7|$
c) $|-2| \cdot (-2)$
d) $\dfrac{-15}{|-5|}$

Exercise 11.4. Compute.

a) $\dfrac{17}{6} - \dfrac{5}{9} - \dfrac{1}{5}$
b) $\dfrac{6}{7} - \dfrac{3}{11} + 1$
c) $\dfrac{5}{8} - \left(\dfrac{3}{4} - \dfrac{3}{5}\right)$
d) $\dfrac{7}{10} - \left(\dfrac{3}{2} + \dfrac{1}{6}\right)$
e) $1 - \left(\dfrac{5}{9} + \dfrac{1}{5}\right)$
f) $\dfrac{5}{6} - \dfrac{1}{3} - \left(2 - \dfrac{13}{9}\right)$

Exercise 11.5. Compute and simplify if needed.

a) $\frac{3}{6}-\left(\frac{3}{4}-\frac{2}{3}\right)$ b) $\left(\frac{5}{4}-\frac{9}{16}\right)-\left(\frac{5}{8}-\frac{11}{12}\right)$

c) $\left(\frac{12}{5}:13\right)\left(\frac{5}{3}-3\right)$ d) $\dfrac{2+\frac{7}{3}}{4-\frac{11}{3}}$

e) $\frac{10}{7}\cdot\frac{4}{15}+\frac{7}{6}\cdot\frac{5}{28}$ f) $\frac{13}{27}\cdot\frac{18}{5}+\frac{13}{210}\cdot\frac{28}{65}$

g) $\left(\frac{1}{2}+\frac{1}{3}-\frac{5}{7}\right):\frac{5}{6}$ h) $15-3\left(-\frac{5}{6}+\frac{3}{4}\right)\cdot 6$

i) $\left(\frac{2}{3}-\frac{3}{4}\right)\cdot\left(\frac{4}{5}+\frac{8}{20}\right)$ j) $\left(\frac{2}{3}-\frac{3}{4}\right)\cdot\frac{4}{5}+\frac{8}{20}$

Exercise 11.6. André plays darts. He shoots 12 darts and hits 10 times the target. Gérard shoots 8 darts and hits 5 times the target. Which one is more skillful?

Exercise 11.7. In Mr. Toubon's class, 3 out of 21 students are sick on Thursdays. On the same day, in Mrs. Robiot's class, 4 out of 20 students are sick. Which class has the highest sick ratio?

Exercise 11.8. Manuel buys a home cinema on sale and pays 567 francs. Calculate the price he would have paid if he hadn't had a 10% discount.

Exercise 11.9. When he goes to his cardiologist, a patient pays 23 francs for the consultation. 70% of this amount is refunded to him by social security. Of the amount remaining after reimbursement by the social security, his mutual insurance company reimburses him 80%. What percentage of the consultation fee did he end up paying?

Exercise 11.10. Compute without calculator.

a) $\frac{9^3}{9^5}$ b) $(-6)^3\cdot(-6)^4$

c) $(-3)^7\cdot(-3)^8$ d) $2^{-3}\cdot(2^{11}:2^4)$

e) $5^3\cdot(5^2)^3$ f) $\frac{2^2\cdot 4^2}{2^3}$

Exercise 11.11. Commpute without calculator.

a) $-\left(\frac{5}{2}\right)^{-1}$ b) $-\left(\frac{5}{2}\right)^{-2}$ c) $\left(-\frac{5}{2}\right)^{2}$

Exercise 11.12. Compute without calculator.

a) $\sqrt[3]{-8}$ b) $\sqrt[6]{1}$

c) $\sqrt[4]{81}$ d) $\sqrt[5]{32}$

e) $\sqrt[3]{0,000001}$ f) $\sqrt[3]{0,027}$

11.2 Elementary algebra

Exercise 11.13. Compute and reduce the like terms.

a) $-4a^4 \cdot (2a - 3b^2)$ b) $-2x^2y(4xy + 2x^2y - x + 2)$
c) $(x + 3) \cdot (x - 2)$ d) $(2x - 5) \cdot (3y - 7)$
e) $3xy - 6xz + yz + 5xy + 6xz - 3yz - (-3xy + xz + 3yz)$ f) $15t - [(3t - 6u) - v] - [5u - (20t + 4v)]$
g) $5x - \{3x - [4y - (8x - 5y) + 4x]\} - 4y$ h) $7x \cdot (x - y) \cdot (x + y)$
i) $-3 + 2xy(x - 2xy) - \left\{-\frac{1}{2}x^2y - (xy + 2x)\right\}$ j) $5x^3 - x\{4 + 3[4 - 5(x - 3)]\}$
k) $(2x + 3) \cdot (x^2 + x - 1) - x^2 - (x^2 + 1) \cdot (x + 4)$ l) $2a \cdot (3a^2 - a) - 3b^3 \cdot (2ab - 8)$

Exercise 11.14. Compute using the remarkable identities.

a) $(3x - 2)^2$ b) $(3x - 2) \cdot (3x + 2)$
c) $(3x - 2y)^2$ d) $(7x - 1)^2$
e) $(7x - 1) \cdot (7x + 1)$ f) $(4x^3 + 2)^2$
g) $(a + 1)(a - 1)^2$ h) $(2 - a)^2 - (a + 2)(2 - a) + (a + 2)^2$

Exercise 11.15. Factor out common factors.

a) $24xy + 8x$ b) $8ac^2 - 24bc^2$
c) $2rny - 4ny + 6y$ d) $xyz + x^2yz - xy^2z + 2xyz^2$
e) $4a^2b - 2a^3c + 6av$ f) $m(a - b) + n(a - b)$
g) $2(x + 1)^2 + 4(x + 1)$ h) $(a + b)^3 + (a + b)^2$

Exercise 11.16. Factorize using remarkable identities.

a) $x^2 + 2xy + y^2$ b) $x^2 + 2x + 1$
c) $x^2 - 144$ d) $9x^2 + 24x + 16$
e) $100a^2 - 9b^2$ f) $x^2 + xy + \frac{y^2}{4}$

Exercise 11.17. Factorize the following trinomials.

a) $x^2 + 8x + 15$ b) $x^2 + 3x - 28$

Exercise 11.18. Compute and simplify if needed.

a) $\frac{10x^2y^5}{5x^5y^2}$ b) $\frac{45(x + 1)}{63(x^2 - 1)}$
c) $\frac{a^2b - ab}{a - 1}$ d) $\frac{x^2 - 4x + 4}{x^2 - 4}$
e) $\frac{a^2 - 2a + 1}{a^4 - 1}$ f) $\frac{(x - y)^2}{3a} : \frac{x^2 - y^2}{6a}$
g) $\frac{x^2 - 1}{x + 2} : \frac{x^2 - 3x + 2}{x + 2}$ h) $\frac{a - b}{6} + \frac{a + 2b}{9}$
i) $\frac{2x - 1}{3} + \frac{x - 2}{15} + \frac{3x + 1}{5}$ j) $\frac{5x - 3b}{3a} - \frac{2y + a}{5b}$
k) $\frac{4}{(5s - 2)^2} + \frac{s}{5s - 2}$ l) $\frac{x}{x^2 - 1} + \frac{1}{x - 1} - \frac{2}{x + 1} + 3$

11.3 Equations

Exercise 11.19. Solve.

a) $\dfrac{x}{3}+5=2$

b) $\dfrac{1}{3}x(x+2)=2+\dfrac{1}{3}x^2$

c) $(x+3)^2=(x+4)^2-(2x+7)$

d) $\dfrac{x}{5}-\dfrac{x}{3}+\dfrac{3x}{10}+\dfrac{7x}{2}=\dfrac{4x}{15}$

e) $(2x+1)^2+(3x+1)^2+(8x-3)^2=(7x-2)(11x-1)$

f) $3(x+5)=\dfrac{x-1}{4}$

g) $\dfrac{x(x+2)}{3}=\dfrac{x^2}{3}+2(x+1)$

h) $\dfrac{1}{2}(3x-1)-\dfrac{1}{4}(4-x)=0$

i) $\dfrac{x+3}{4}=\dfrac{9}{5}-\dfrac{x+3}{5}$

j) $3x-\dfrac{1}{2}(x+5)=5-\dfrac{x-3}{6}$

k) $\dfrac{3x+2}{5}-x-\dfrac{2x+5}{3}=3$

l) $5+\dfrac{1}{2}(11x-37)=\dfrac{31}{10}x+\dfrac{2}{5}(6x-40)+\dfrac{5}{2}$

Exercise 11.20. Solve the following systems using the method that seems most appropriate.

a) $\begin{cases} x+5y &= 47 \\ x+y &= 15 \end{cases}$

b) $\begin{cases} 7x-3y &= 0 \\ x+y &= 50 \end{cases}$

c) $\begin{cases} 3x+7y &= 1 \\ y-3x &= 7 \end{cases}$

d) $\begin{cases} 3x-y &= 4 \\ 5x-2y &= 5 \end{cases}$

e) $\begin{cases} \dfrac{x+2y}{2} &= \dfrac{y+79}{4} \\ \dfrac{33x+13y}{12}-y &= \dfrac{x}{2} \end{cases}$

f) $\begin{cases} \dfrac{5}{4}x-\dfrac{1}{2}y+\dfrac{1}{4} &= 0 \\ -x-\dfrac{7-3y}{2} &= 0 \end{cases}$

g) $\begin{cases} \dfrac{x}{12}-\dfrac{1-2y}{4} &= \dfrac{1}{6} \\ \dfrac{29}{10}-2+x &= -\dfrac{y}{10} \end{cases}$

h) $\begin{cases} -\dfrac{x+y}{9} &= \dfrac{1}{27}(y+2) \\ \dfrac{-x+1}{10}-\dfrac{y}{6}+\dfrac{1}{5} &= \dfrac{8}{15} \end{cases}$

Exercise 11.21. Solve.

a) $x^2-6x+5=0$

b) $4x^2+4x=-1$

c) $x(x-8)+7=0$

d) $\dfrac{9}{x}-\dfrac{x}{3}=2$

e) $(x+4)^2-x-2-3x^2=0$

f) $(x+1)(x+2)=(x+2)(x-3)-(2x-1)^2$

Exercise 11.22. Solve.

a) $x^4-25x^2+144=0$

b) $x^4+5x^2-36=0$

c) $x^2+\dfrac{18}{x^2}-11=0$

d) $x^6+2x^3+1=0$

Exercise 11.23. Solve.

a) $\sqrt{x^2-1}=\sqrt{3-x^2}$

b) $x-1+\sqrt{x}=1$

c) $x+\sqrt{5x+10}=8$

d) $\sqrt{x-2}+\dfrac{1}{\sqrt{x-2}}=2$

Exercise 11.24. We buy seven identical boxes of toffees. If you pay with a 10-franc note, you get 1,60 francs back. How much does a box of toffees cost?

Exercise 11.25. Find a number such that the sum of its quotients by two, by three and by four is equal to 130.

Exercise 11.26. How many litres of wine at 6 francs per litre have to be mixed with wine at 9 francs per litre to make 60 litres of wine worth 480 francs?

Exercise 11.27. The flight from Los Angeles to Albuquerque, with a layover in Phoenix, costs 90 francs from Los Angeles to Phoenix and 120 francs from Los Angeles to Albuquerque. A total of 185 passengers boarded the plane in Los Angeles, and the total revenue was 21'000 francs. How many passengers got off the plane in Phoenix?

Exercise 11.28. Two years ago, the sum of the ages of a mother and her daughter was 90. 12 years ago the mother was 2 years younger than twice the age of her daughter. What are the ages of these 2 people today?

Exercise 11.29. Consider a rectangle of width x and length y. If we increase the width by 3 cm and decrease the length by 2 cm, its area remains unchanged. It is equally true if we decrease the width by 2 cm and increase the length by 3 cm. What are the dimensions of this rectangle?

Exercise 11.30. A jackpot must be shared among the members of a society that organizes a race. Four people cannot take part in the race, which increases the share of the others by 50 francs. How many people are there in the society if the jackpot is worth 4'000 francs?

11.4 Functions

Exercise 11.31. This is a table of values representing a function f.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	5	2	1	-3	-4	5	3	4	-4

a) What is the image of 3 under f?

b) What number has an image of -3 under f?

c) What numbers have the same image under f?

Exercise 11.32. Let f be a function defined by $f(x) = 4 - x^2$. Compute

a) $f(0)$ b) $f(\sqrt{2})$

c) $3f(x)$ d) $f(3x)$

e) $f(-x)$ f) $f(\sqrt{2}-2)$

Exercise 11.33. Graphically represent the following functions.

a) $f(x) = 2x$ b) $f(x) = -2 + \dfrac{4}{3}x$

11.5 First degree functions

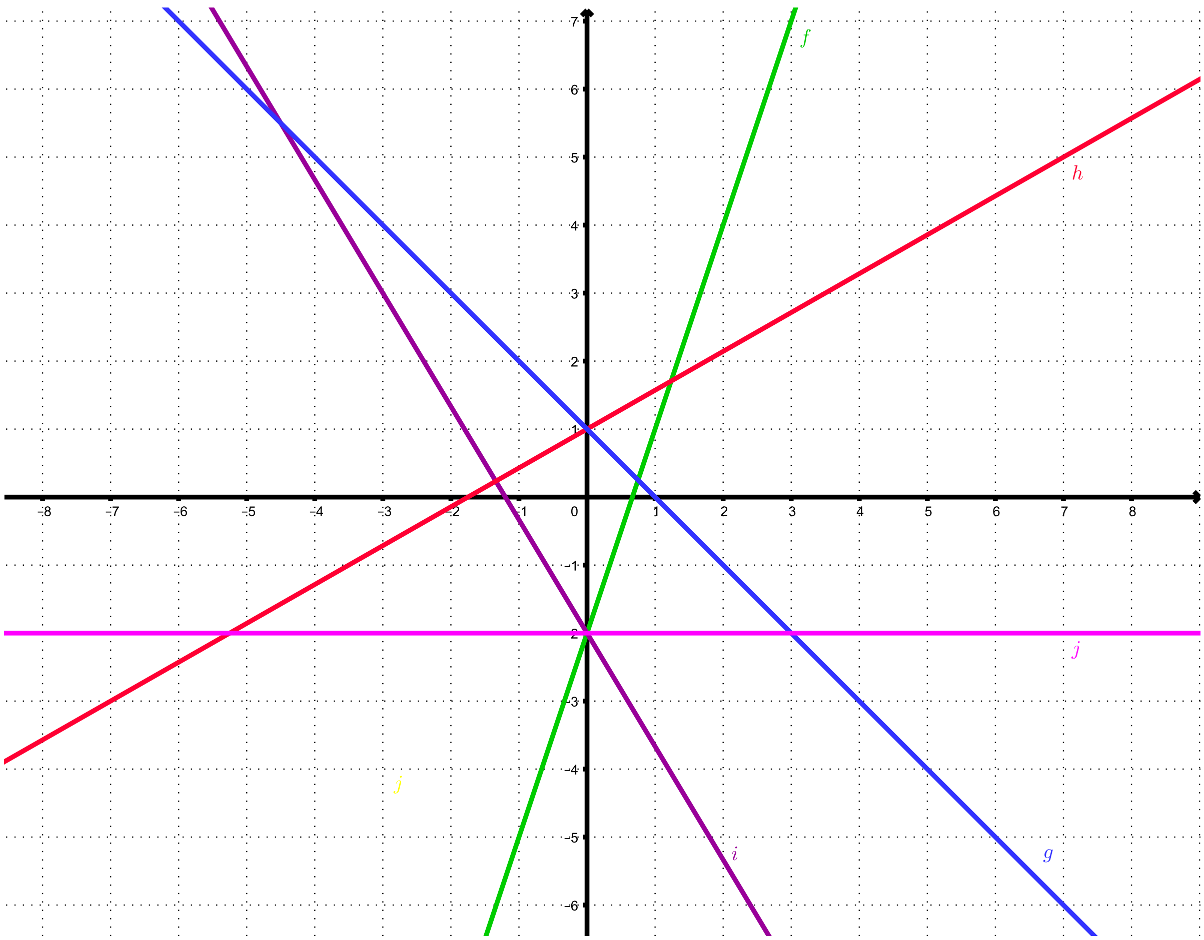
Exercise 11.34. Sketch the following lines on a Cartesian plan.

a) $y = \frac{5}{3}x - 2$ b) $y = -\frac{1}{2}x + 3$

c) $6x - 2y + 4 = 0$ d) $5x + 2y = 2$

Exercise 11.35. Determine the functional notation of the function f that passes through the points $A(5; 1)$ and $B(1; -4)$.

Exercise 11.36. Determine the equation of each lines.

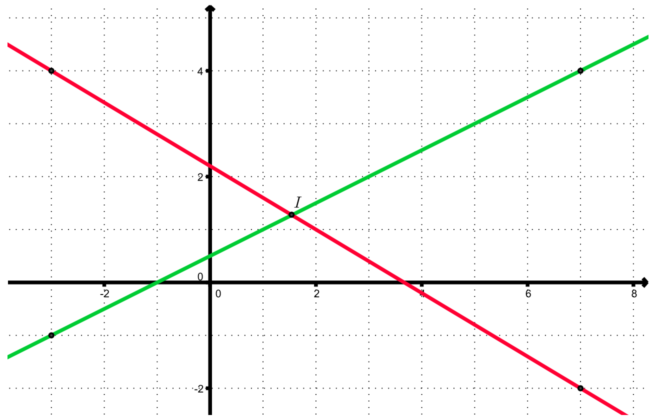


Exercise 11.37. Find an equation of the line passing through the point $P(3; -5)$ perpendicular to the line $3x + 2y = 6$.

Exercise 11.38. Determine the coordinates of the intersection point of functions f and g.

$$f(x) = \frac{3}{5}x - 2 \text{ and } g(x) = \frac{1}{2}x + 1$$

Exercise 11.39. Let's consider this figure.



a) Calculate the coordinates of the point of intersection I of the two lines drawn above.

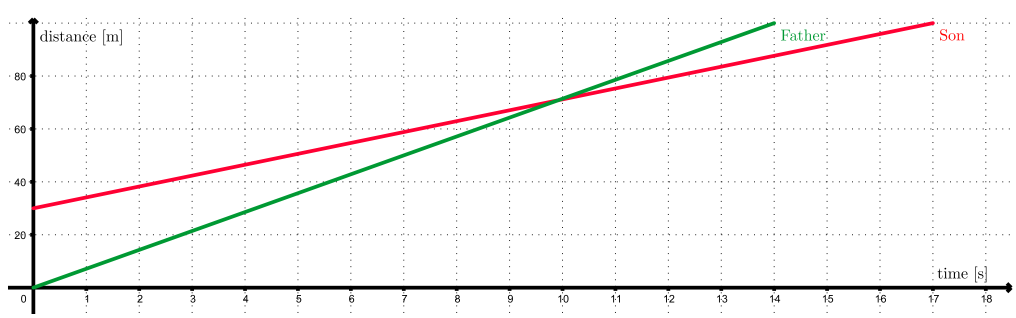
b) Find the function f whose graph is a line passing through the origin and the point I.

c) Find the function g whose graph is a line parallel to the graph of f and passes through the point $P(2; -1)$.

Exercise 11.40. Give the coordinates of the points of intersection of each of the following lines with the axes.

a) $y = 4x - 8$ b) $3x - 2y = 6$

Exercise 11.41. A father challenges his son at a 100 m sprint and leaves him a certain number of meters ahead. Simplified graphs of this race are given below.



a) How many meters ahead does the father let the son have?

b) Who won? How many seconds early?

c) How far apart are they when the winner crosses the finish line?

d) What's the father's speed, the son's speed (in m/s)?

e) What does the slope of each line actually correspond to?

f) Were father and son side by side? If so, how many meters did the father cover at that time?

Exercise 11.42. A written assignment contains 17 points. A grade of 1 corresponds to 0 point and a grade of 6 to 17 points. Determine the function to calculate the grade y as a function of the number of points x.

Exercise 11.43. In the village of Courvelier, each citizen receives a bill for his or her water consumption:

— an annual fixed fee of 40 francs;

— 2,5 francs per m^3 of consumed water.

a) Express the function that describes, in this village, the annual bill y as a function of the consumption x (in m^3) .

b) In 2017, Noah, inhabitant of Courvelier, is a determined sportsman and regularly showers in gyms. Therefore, his consumption at home is rather low. In the past year, Noah consumed 65 m^3 of water. How much was his bill?

Mia lives in Montébert. She has a stressful job and enjoys taking a bath regularly, which makes her water consumption rather high. In 2016, Mia paid a bill of 270 francs for 100 m^3 of water, while in 2017, her water bill was 253 francs for 90 m^3 of water consumed.

c) Express the function that describes the annual bill as a function of consumed water in Montébert.

d) Assuming that Noah and Mia use the same amount of water over a year, for what consumption will they pay the same bill?

e) Courvelier and Montébert are considering to merge. With regard to water consumption, these municipalities are asking their citizens for their opinion on the water rate to be introduced. Which municipal rate would Noah prefer? What about Mia?

Exercise 11.44. A theatre offers the three options below:

— 20 francs per performance.

— A 48 franc privilege card that gives you a 30% discount on performance.

— An annual subscription of 300 francs that allows free entry to each performance.

a) Express the functions that describe the annual bill y as a function of the number of performances you attend for each option.

b) Eddy prefers comedy. He thinks he'll attend 20 in a year. How much will he pay if he choose option 2? In this situation, would another option be better?

c) From how many and up to how many performances is option 2 the most advantageous?

11.6 Quadratic functions

Exercise 11.45. Find the intersection points of the following parabolas with the axes and the vertex (determine if it's a maximum or a minimum).

a) $y = x^2 - 5x + 6$	b) $y = -2(x+3)(x-4)$
c) $y = -3(x-2)^2 - 1$	d) $y = x^2 + 9x + 18$
e) $y = -3(x-2)^2$	f) $y = 4x^2 - 4x - 8$

Exercise 11.46. The risk of developing health problems can be evaluated by the function

$$R(i) = 0.002i^2 - 0.08i + 0.85$$

where i is the body mass index. This index is expressed by the following relationship:

$$i = \frac{\text{Mass in kg}}{(\text{Height in meters})^2}.$$

A person is 1,75 m high and weighs 80 kg.

a) What is his body mass index?

b) What's his ideal weight?

Exercise 11.47. Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be parabolas of equations

$$\mathcal{P}_1 : y = x^2 + x - 6;$$

$$\mathcal{P}_2 : y = -x^2 + x + 2.$$

Compute

a) The coordinates of the intersection point of $\mathcal{P}_1$ with the x-axis.

b) The coordinates of the intersection point of $\mathcal{P}_2$ with the x-axis.

c) The coordinates of the intersection point of $\mathcal{P}_1$ with the y-axis.

d) The coordinates of the intersection point of $\mathcal{P}_2$ with the y-axis.

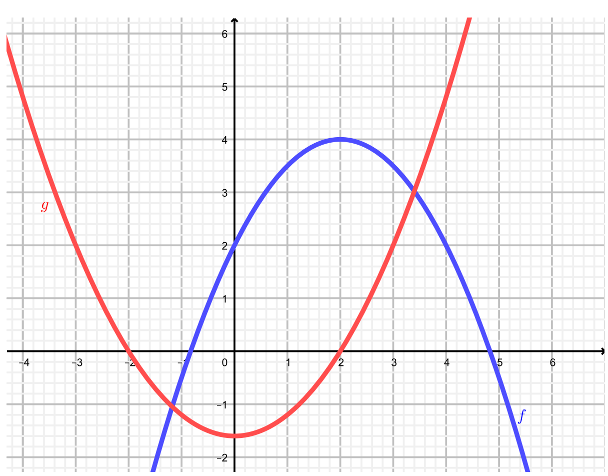
e) The coordinates of the vertex of $\mathcal{P}_1$.

f) The coordinates of the vertex of $\mathcal{P}_2$.

g) The coordinates of the intersection points of $\mathcal{P}_1$ and $\mathcal{P}_2$.

Exercise 11.48. Determine the equation of the parabola whose vertex is $V(-6;-3)$ and passing through point $A(5;-1)$.

Exercise 11.49. Determine the polynomial form of each of the following parabolas.



Exercise 11.50. What is the maximum value of a product of two numbers if their sum is equal to 35?

Exercise 11.51. In a bowling alley, people normally play 3000 games per day at the price of 2 USD each. Assuming that each reduction of 5 cents let increase of 100 the number of the played games, find the price to be set for a maximum income and the value of this maximum income?

Exercise 11.52. A tennis match is renowned expected in a stadium that can accommodate up to 6300 people. A previous study showed that the number of spectators x is based on the price of the entrance ticket p according to the relation

$$x = -50p + 6300,\ p \text{ is in USD.}$$

a) Find the ticket price if the number of spectators is 4’800.

b) Find the equation of the match income.

c) What’s the ticket price corresponding to a maximum income?

Exercise 11.53. The Rolega watchmaking company will launch a new watch model, the Rowatch. To produce the Rowatch, the company has to make an initial investment of one million francs, plus 80 francs for each Rowatch produced. A market study has shown that the demand x for Rowatch based on the selling price p is given by

$$x = 80'000 - 200p.$$

.

a) What is the selling price to the public that ensures maximum profit?

b) How many Rowatch would be sold at that price?

c) What is the maximum benefit that the Rolega company can earn from the sale of the Rowatch?

d) After one year Rolega plans to liquidate unsold Rowatches at the lower break-even price. At what price (rounded up to 5 cents) will the last Rowatch stock be sold?

11.7 Exponential and logarithmic functions

Exercise 11.54. Solve.

a) $5^{x+\frac{1}{4}} = 25$ b) $3^{x^2+2} = 9^{x^2-x+5}$

c) $25^x = 125^{x-2}$ d) $\left(3^{x-1}\right)^3 = 9 \cdot 3^{x-2}$

Exercise 11.55. Compute the following logarithms.

a) $\log_{10}(1'000'000'000)$ b) $\log_8(1)$

c) $\log_{11}\left(\frac{1}{121}\right)$ d) $\log_4(2)$

Exercise 11.56. Solve.

a) $\log_{10}(3x-7) = \log_{10}(5x-9)$ b) $\log_3(x-6) = 2$

c) $\log_9(x) = \frac{1}{2}$ d) $\log_{10}(x^2-7) = 2 \cdot \log_{10}(x+3)$

Exercise 11.57. A biologist knows that a city's dog population grows exponentially. A survey done five years ago shows that there were 2'800 dogs at that time. Today, there are 5'160 dogs. How many dogs will there be in this city in ten years?

Exercise 11.58. A pond contained 1'000 trouts three months ago. It is estimated that there are currently 600 left. How long before there will be less than 150 trouts?

Exercise 11.59. FEDERALPHONE, a long-established state-owned mobile phone operator, has seen its customer base decline from 12'212'000 subscribers in 2000 to 8'962'000 subscribers in 2005. The MOONRISE group, which entered the market at the end of the 1990s, increased its customer base from 4'500'000 to 6'924'000 over the same period.

a) Determine, in %, the annual variation rate of the costumer base for both operators.

b) In which year is it expected that the number of MOONRISE customers will be greater than the number of FEDERALPHONE subscribers?

Exercise 11.60. If 10 g of salt is added to water, then the quantity $q(t)$ of salt (in g) that is not dissolved after t minutes is given by the formula

$$q(t) = 10 \cdot 0,8^t.$$

How much is left after 5 minutes? Estimate how long it takes for 9g of the salt to dissolve.

11.8 Inequations

Exercise 11.61. Solve the following inequalities and give the solution set in interval form :

a) $x + 5(x+1) > 4(2-x) + 2$

b) $-\dfrac{2}{3}x + 3 \geq 0$

c) $\dfrac{15x-4}{2} < 1 + 6x$

d) $\dfrac{7}{2}x - \dfrac{x-1}{2} \leq x + \dfrac{3}{4}$

e) $1 < \dfrac{2}{3}x + 1 < 2$

f) $4 > \dfrac{2-3x}{7} \geq -2$

Exercise 11.62. Solve the following double inequalities and give the solution set in interval form :

a) $\left\{ \begin{array}{rcl} (x-1)(x-2) & \geq & x^2 - 7 \\ (x-2)(x-3) & < & x^2 + 1 \end{array} \right.$

b) $\left\{ \begin{array}{rcl} 2x - 1 & > & x + 3 \\ \dfrac{4x}{3} + 3 & < & \dfrac{x+7}{2} \end{array} \right.$

Exercise 11.63. Graphically solve the following system of inequalities.

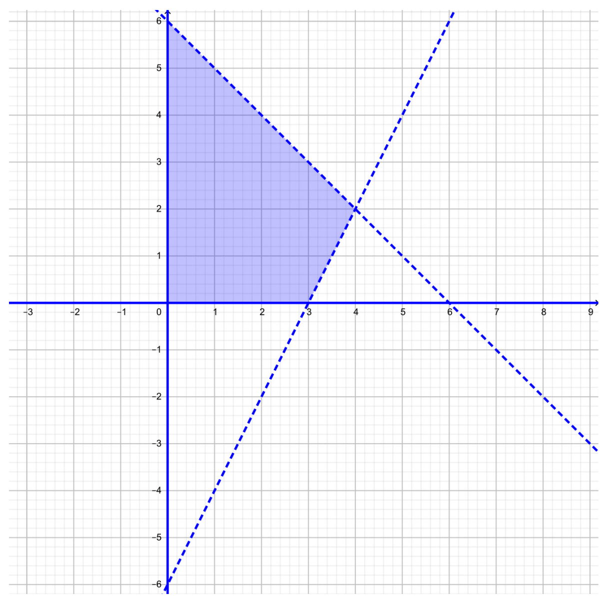
a) $\left\{ \begin{array}{rcl} x - 3y + 6 & > & 0 \\ 3x - 2y + 11 & < & 0 \\ 2x - y - 9 & \leq & 0 \end{array} \right.$

b) $\left\{ \begin{array}{rcl} 3x + 5y & < & 15 \\ y - x & \leq & 0 \end{array} \right.$

c) $\left\{ \begin{array}{rcl} x - 3y + 15 & > & 0 \\ 2x + 3y - 12 & < & 0 \end{array} \right.$

d) $\left\{ \begin{array}{rcl} x + 4y & \leq & 8 \\ x - y & \geq & 3 \\ y & \geq & 0 \end{array} \right.$

Exercise 11.64. Determine the system of inequalities whose solution set is represented below.



11.9 Linear programming

Exercise 11.65. A small community wants to acquire second-hand vans and small buses for its public transportation system. The community cannot spend more than 200'000 francs on the vehicles and no more than 1'000 francs per month on maintenance. The vans cost 20'000 francs each and an average of 200 francs per month for maintenance. The corresponding approximate costs for each bus are 40'000 francs and 150 francs per month. Knowing that each van can carry 15 passengers and each bus 25 passengers, find the number of vans and buses to be purchased to maximize the passenger capacity of the system.

Exercise 11.66. A radio manufacturer makes a profit of 25 francs on the luxury model and 30 francs on the ordinary model. The company wants to produce at least 80 luxury models and 100 ordinary models every day. To maintain high quality, daily production should not exceed 200 radios. How many radios of each type would have to be produced every day in order to make a maximum profit?

Exercise 11.67. An investor's club has 30'000 francs to invest in two projects A and B. The investment in each project can only be made by parts of 1'000 francs. The rate of return for the project A is around 12% and the rate of return for the project B is 7%. The club's policy is to invest at least twice as much in the project B which is less risky than the project A. How many parts does the club need to invest in each project to maximize its expected profit?

11.10 Statistics

Exercise 11.68. Children in three classes were asked what their favourite winter sport was. The following raw data was obtained:

Ice hockey	Sliding	Ice hockey	Ice hockey	Ice hockey
Ice hockey	Skiing	Ice hockey	Skiing	Snowshoe
Ice skating	Skiing	Skiing	Ice hockey	Skiing
Skiing	Ice hockey	Skiing	Snowshoe	Skiing
Ice skating	Skiing	Ice hockey	Snowshoe	Snowshoe
Skiing	Sliding	Ice hockey	Sliding	Sliding
Ice hockey	Sliding	Ice hockey	Ice hockey	Ice hockey
Cross-country skiing	Ice hockey	Ice skating	Ice skating	Ice hockey
Skiing	Ice hockey	Skiing	Snowshoe	Ice skating
Ice hockey	Sliding	Skiing	Skiing	Cross-country skiing
Ice hockey	Ice skating	Skiing	Ice skating	Ice hockey
Ice hockey	Ice skating	Skiing	Ice skating	Snowshoe

a) Identity the population.

b) Identify the variable. What's its type?

c) Give all the outcomes.

d) Create a table with the frequencies and the percents and a pie chart.

Exercise 11.69. This dataset represent the age of 50 tennis players from a club.

33	30	19	29	17	31	33	29	30	28
32	22	27	27	23	31	28	26	20	34
30	35	27	32	26	29	27	22	28	18
23	31	36	24	27	30	35	31	40	31
41	25	30	38	25	29	36	28	39	28

a) Arrange this data into 5 classes of equal size, using $16,5$ as the lower bound. Then calculate the relative frequency and the cumulative relative frequency.

b) Sketch the histogram and the relative frequency polygon.

c) Sketch the cumulative relative frequency polygon.

Exercise 11.70. We measured the height of 50 teachers of a school.

Height in cm	Number of teachers
$[130; 140[$	2
$[140; 150[$	4
$[150; 160[$	7
$[160; 170[$	8
$[170; 180[$	15
$[180; 190[$	10
$[190; 200[$	4

Compute the mean, determine the median class and the modal class.

Exercise 11.71. In Canada, rainbow trout have been caught and measured for their length (in cm) before release in water. The following data set is obtained.

Length	Frequency
$]10; 20]$	12
$]20; 25]$	28
$]25; 30]$	41
$]30; 35]$	22
$]35; 40]$	18

a) Draw the histogram of this set, as well as the polygons of the increasing and increasing cumulative frequencies.

b) Determine the modal class, the median class and compute the arithmetic mean.

c) Compute the variance and the standard deviation.

d) Measurements taken one year earlier resulted in a mean of 32 cm with a standard deviation of 7 cm. Which of the two populations has the greatest dispersion?

11.11 Solutions

Exercise 11.1.

a) 39	b) 0
c) 44	d) 9
e) 31	f) 104
g) 480	h) 131
i) 211	j) 17
k) 188	l) 39
m) 16	n) 51
o) 14	p) 12

Exercise 11.2.

a) 0	b) 319
c) -24	d) 8
e) 12	f) Impossible
g) 2	h) Undetermined
i) 3	j) 25

Exercise 11.3.

a) -5	b) 10
c) -4	d) -3

Exercise 11.4.

a) $\frac{187}{90}$	b) $\frac{122}{77}$
c) $\frac{19}{40}$	d) $-\frac{29}{30}$
e) $\frac{11}{45}$	f) $-\frac{1}{18}$

Exercise 11.5.

a) $\frac{5}{12}$	b) $\frac{47}{48}$
c) $-\frac{16}{65}$	d) 13
e) $\frac{33}{56}$	f) $\frac{44}{25}$
g) $\frac{1}{7}$	h) $\frac{33}{2}$
i) $-\frac{1}{10}$	j) $\frac{1}{3}$

Exercise 11.6. Andre was the most skillful.

Exercise 11.7. Mrs. Robiot's class.

Exercise 11.8. 630 francs.

Exercise 11.9. He paid 6% of the price.

Exercise 11.10.

a) 9^{-2} b) $(-6)^7 = -6^7$
c) $(-3)^{15} = -3^{15}$ d) 2^4
e) 5^9 f) 2^3

Exercise 11.11.

a) $-\dfrac{2}{5}$ b) $-\dfrac{4}{25}$ c) $\dfrac{25}{4}$

Exercise 11.12.

a) -2 b) 1
c) 3 d) 2
e) $0,01$ f) $0,3$

Exercise 11.13.

a) $-8a^5 + 12a^4b^2$ b) $-8x^3y^2 - 4x^4y^2 + 2x^3y - 4x^2y$
c) $x^2 + x - 6$ d) $6xy - 14x - 15y + 35$
e) $11xy - xz - 5yz$ f) $32t + u + 5v$
g) $5y - 2x$ h) $7x^3 - 7xy^2$
i) $-4x^2y^2 + \dfrac{5}{2}x^2y + xy + 2x - 3$ j) $5x^3 + 15x^2 - 61x$
k) $x^3 - 7$ l) $6a^3 - 2a^2 - 6ab^4 + 24b^3$

Exercise 11.14.

a) $9x^2 - 12x + 4$ b) $9x^2 - 4$
c) $9x^2 - 12xy + 4y^2$ d) $49x^2 - 14x + 1$
e) $49x^2 - 1$ f) $16x^6 + 16x^3 + 4$
g) $a^3 - a^2 - a + 1$ h) $3a^2 + 4$

Exercise 11.15.

a) $8x(3y + 1)$ b) $8c^2(a - 3b)$
c) $2y(rn - 2n + 3)$ d) $xyz(1 + x - y + 2z)$
e) $2a(2ab - a^2c + 3v)$ f) $(a - b)(m + n)$
g) $2(x + 1)(x + 3)$ h) $(a + b)^2(a + b + 1)$

Exercise 11.16.

a) $(x + y)^2$ b) $(x + 1)^2$
c) $(x + 12)(x - 12)$ d) $(3x + 4)^2$
e) $(10a + 3b)(10a - 3b)$ f) $\left(x + \dfrac{y}{2}\right)^2$

Exercise 11.17.

a) $(x+3)(x+5)$ b) $(x+7)(x-4)$

Exercise 11.18.

a) $\dfrac{2y^3}{x^3}$ b) $\dfrac{5}{7(x-1)}$

c) ab d) $\dfrac{x-2}{x+2}$

e) $\dfrac{a-1}{(a^2+1)(a+1)}$ f) $\dfrac{2(x-y)}{x+y}$

g) $\dfrac{x+1}{x-2}$ h) $\dfrac{5a+b}{18}$

i) $\dfrac{20x-4}{15}$ j) $\dfrac{25bx-15b^2-6ay-3a^2}{15ab}$

k) $\dfrac{5s^2-2s+4}{(5s-2)^2}$ l) $\dfrac{3x^2}{x^2-1}$

Exercise 11.19.

a) $x=-9$ b) $x=3$

c) Infinity of solutions d) $x=0$

e) $x=1$ f) $x=-\dfrac{61}{11}$

g) $x=-\dfrac{3}{2}$ h) $x=\dfrac{6}{7}$

i) $x=1$ j) $x=3$

k) $x=-4$ l) Infinity of solutions

Exercise 11.20.

a) $(x;y)=(7;8)$ b) $(x;y)=(15;35)$

c) $(x;y)=(-2;1)$ d) $(x;y)=(3;5)$

e) $(x;y)=(-1;27)$ f) $(x;y)=(1;3)$

g) $(x;y)=(-1;1)$ h) $(x;y)=(6;-5)$

Exercise 11.21.

a) $x=1$ and $x=5$ b) $x=-\dfrac{1}{2}$

c) $x=1$ and $x=7$ d) $x=-9$ and $x=3$

e) $x\cong-1,42$ and $x\cong 4,92$ f) No solution

Exercise 11.22.

a) $x=3, x=-3, x=4$ and $x=-4$ b) $x=2$ and $x=-2$

c) $x=3, x=-3, x=\sqrt{2}$ and $x=-\sqrt{2}$ d) $x=-1$

Exercise 11.23.

a) $x = \sqrt{2}$ and $x = -\sqrt{2}$ b) $x = 1$
c) $x = 3$ d) $x = 3$

Exercise 11.24. $1,20$ francs.

Exercise 11.25. The number is 120.

Exercise 11.26. He must mix 20 litres of wine at 6 francs and 40 litres of wine at 9 francs.

Exercise 11.27. 40 passengers.

Exercise 11.28. 58 and 36 years.

Exercise 11.29. 6 cm and 6 cm.

Exercise 11.30. 20 persons.

Exercise 11.31.

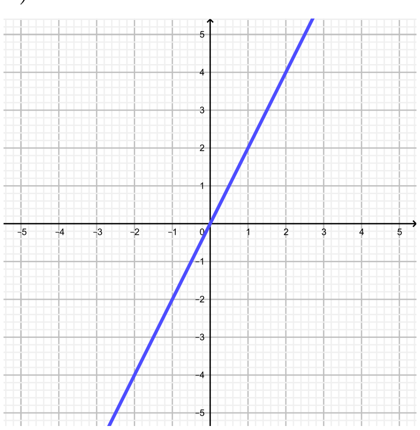
a) 4.
b) -1.
c) 1 and -4; 0 and 4.

Exercise 11.32.

a) 4 b) 2
c) $12 - 3x^2$ d) $4 - 9x^2$
e) $4 - x^2$ f) $4\sqrt{2} - 2$

Exercise 11.33.

a)



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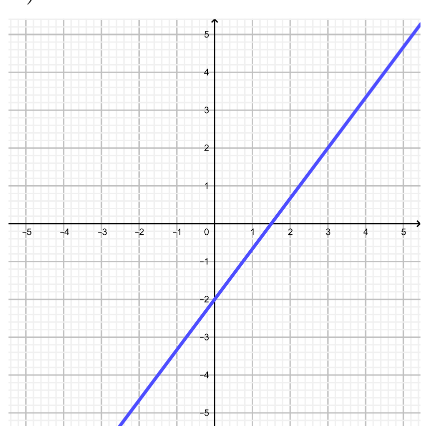
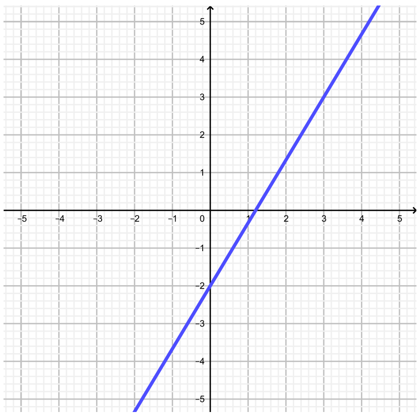


Exercise 11.34.

a)



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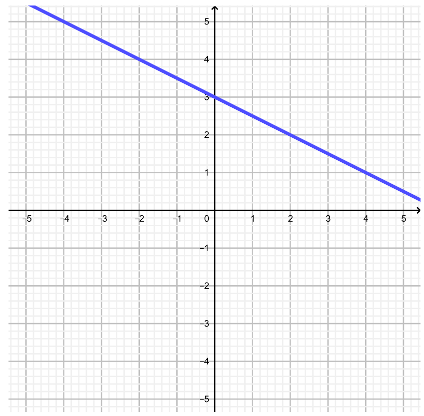

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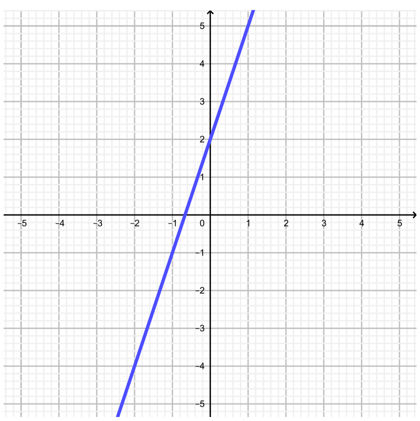

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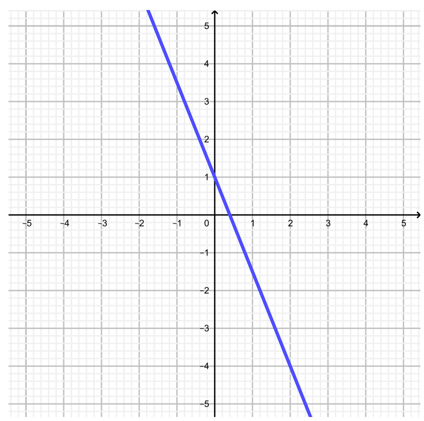


Exercise 11.35. $f(x) = \frac{5}{4}x - \frac{21}{4}$.

Exercise 11.36.

$f(x) = 3x - 2.$

$g(x) = -x + 1.$

$h(x) = \frac{4}{7}x + 1.$

$i(x) = -\frac{5}{3}x - 2.$

$j(x) = -2.$

Exercise 11.37. $y = \frac{2}{3}x - 1.$

Exercise 11.38. $I(30; 16)$

Exercise 11.39.

a) $I\left(\frac{17}{11};\frac{14}{11}\right)$, the two represented functions are $y=-\frac{3}{5}x+\frac{11}{5}$ and $y=\frac{1}{2}x+\frac{1}{2}$.

b) $f(x)=\frac{14}{17}x$.

c) $g(x)=\frac{14}{17}x-\frac{45}{17}$.

Exercise 11.40.

a) $I_1(2;0)$ and $I_2(0;-8)$

b) $I_1(2;0)$ and $I_2(0;-3)$

Exercise 11.41.

a) 30 m.

b) The father won by three seconds.

c) Around $12,35$ m.

d) $s_{\text{father}} \cong 7,14$ m/s and $s_{\text{son}} \cong 4,12$ m/s.

e) It corresponds to the respective speeds.

f) Around $70,83$ m.

Exercise 11.42. $y=\frac{5}{17}x+1$.

Exercise 11.43.

a) $y_C=2,5x+40$.

b) 202,5 francs.

c) $y_M=1,7x+100$.

d) 75 m^3.

e) They will support the rate currently used in their respective villages.

Exercise 11.44.

a) $y_1=20x$, $y_2=14x+48$ and $y_3=300$.

b) With option 2, he will pay 328 francs. The third option is better since he would only pay 300 francs.

c) Between 8 and 18 performances.

Exercise 11.45.

a) $I_1\left(2;0\right), I_2\left(3;0\right), I_3\left(0;6\right)$ and $V\left(\frac{5}{2};-\frac{1}{4}\right)$, minimum

b) $I_1\left(-3;0\right), I_2\left(4;0\right), I_3\left(0;24\right)$ and $V\left(\frac{1}{2};\frac{49}{2}\right)$, maximum

c) $I\left(0;-13\right)$ and $V(2;-1)$, maximum

d) $I_1\left(-6;0\right), I_2\left(-3;0\right), I_3\left(0;18\right)$ and $V\left(-\frac{9}{2};-\frac{9}{4}\right)$, minimum

e) $I_1\left(2;0\right), I_2\left(0;-12\right)$ and $V(2;0)$, maximum

f) $I_1\left(-1;0\right), I_2\left(2;0\right), I_3\left(0;-8\right)$ and $V\left(\frac{1}{2};-9\right)$, minimum

Exercise 11.46.

a) Around 26.12.

b) $61,25$ kg.

Exercise 11.47.

a) $I_1(-3;0)$ and $I_2(2;0)$.

b) $I_1(2;0)$ and $I_2(-1;0)$.

c) $I(0;-6)$.

d) $I(0;2)$.

e) $S\left(-\frac{1}{2};-\frac{25}{4}\right)$.

f) $S\left(\frac{1}{2};\frac{9}{4}\right)$.

g) $I_1(-2;-4)$ and $I_2(2;0)$.

Exercise 11.48. $f(x)=\dfrac{2}{121}(x+6)^2-3$.

Exercise 11.49. $f(x)=-\dfrac{1}{2}x^2+2x+2$ and $g(x)=\dfrac{2}{5}x^2-\dfrac{8}{5}$.

Exercise 11.50. $\dfrac{35}{2}$ and $\dfrac{35}{2}$. The product is therefore $\dfrac{1225}{4}$.

Exercise 11.51. 1,75 francs per match and 6125 of profit.

Exercise 11.52.

a) 30 francs.

b) $R=-50p^2+6'300p$.

c) 63 francs.

Exercise 11.53.

a) 240 francs.

b) 32'000 watches.

c) 4'120'000 francs.

d) 96,45 francs.

Exercise 11.54.

a) $x=\dfrac{7}{4}$

b) No solution

c) $x=6$

d) $x=\dfrac{3}{2}$

Exercise 11.55.

a) 9 b) 0

c) -2 d) $\frac{1}{2}$

Exercise 11.56.

a) No solution ($x = 1$ leads to a negative number for the logarithm) b) $x = 15$

c) $x = 3$ d) $x = -\frac{8}{3}$

Exercise 11.57. $\sim 17'516$ dogs

Exercise 11.58. In 12 months.

Exercise 11.59.

a) Federalphone's customer base is declining at an annual rate of 6% and Moonrise's customer base is growing at an annual rate of 9%.

b) In 2007.

Exercise 11.60. $q(5) \cong 3,28\%$ g. Après 18 minutes.

Exercise 11.61.

a) $x \in \left]\frac{1}{2}; +\infty\right[$ b) $x \in \left]-\infty; \frac{9}{2}\right]$

c) $x \in]-\infty; 2[$ d) $x \in \left]-\infty; \frac{1}{8}\right]$

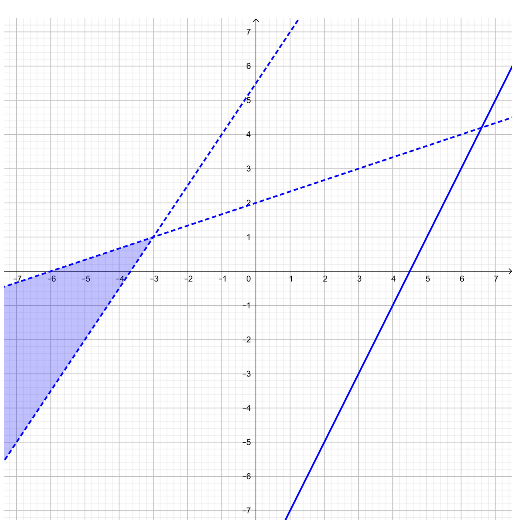
e) $x \in \left]0; \frac{3}{2}\right[$ f) $x \in \left[-\frac{26}{3}; \frac{16}{3}\right[$

Exercise 11.62.

a) $x \in]1; 3]$ b) No solution

Exercise 11.63.

a)



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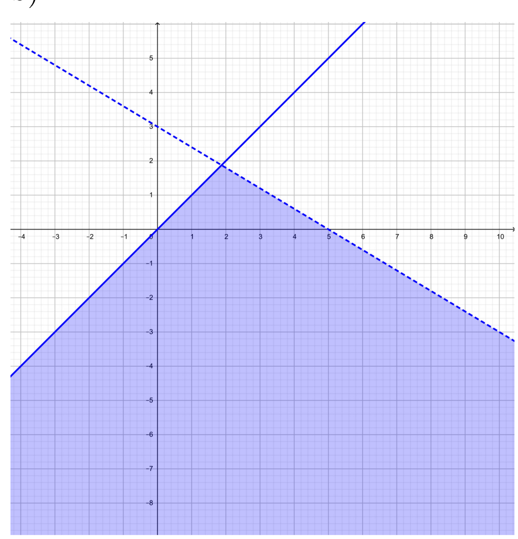

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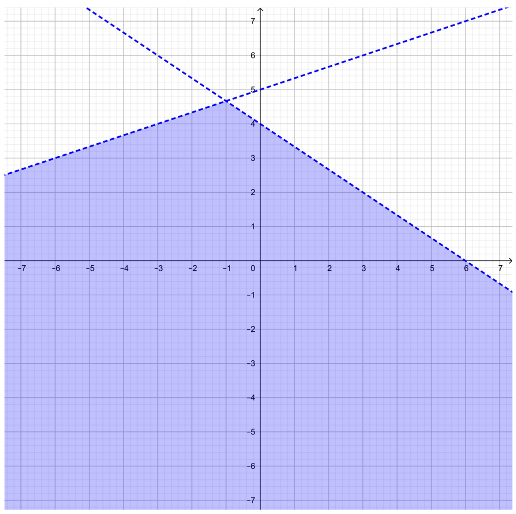

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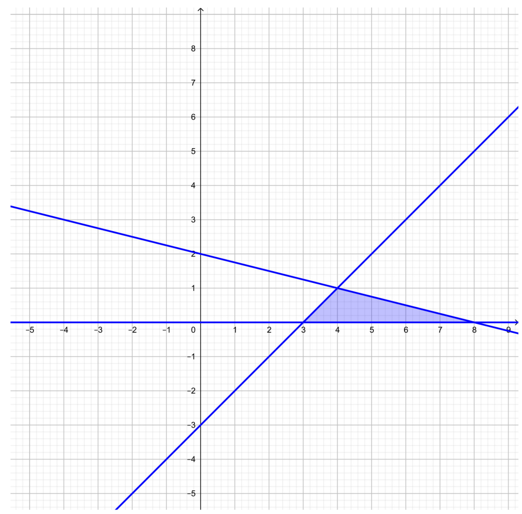


Exercise 11.64.

$$\begin{cases} y > 2x - 6 \\ y < -x + 6 \\ x \geq 0 \\ y \geq 0 \end{cases}.$$

Exercise 11.65. The maximum passenger capacity will be reached with 2 vans and 4 buses for the maximum transport of 130 passengers.

Exercise 11.66. The weekly profit is maximum for a production of 80 luxury models and 120 ordinary models.

Exercise 11.67. The maximum gain will be realized with 10 parts in the project A and 20 parts in the project B.

Exercise 11.68.

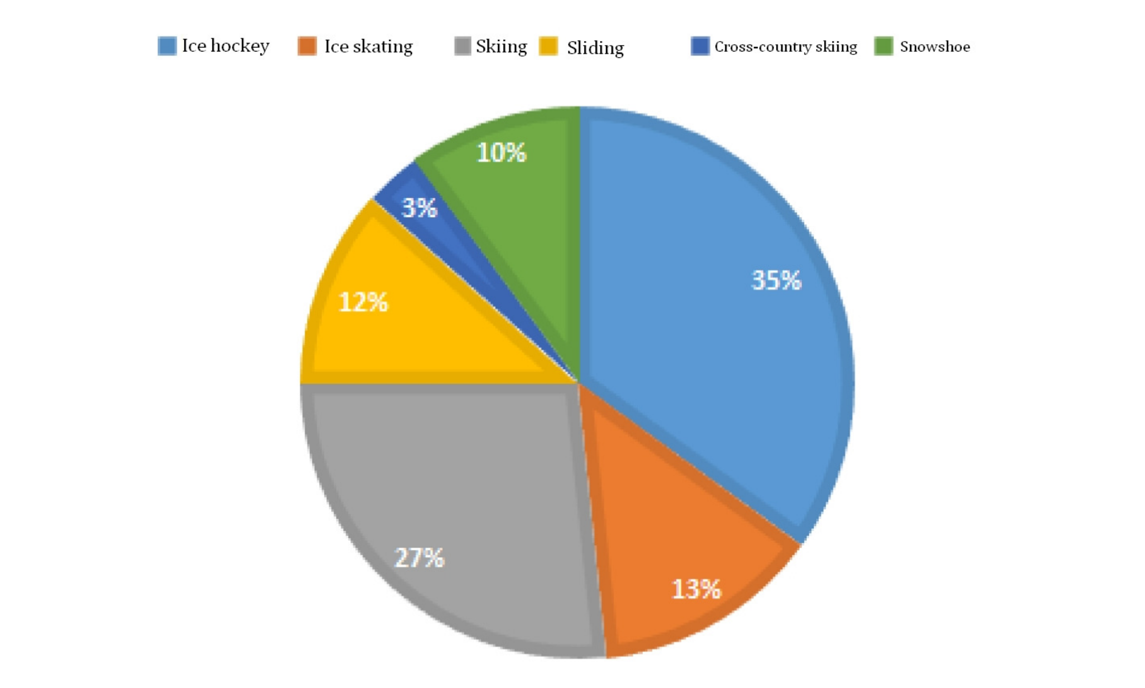
a) Three classes of a school.

b) Their favorite sport. It's a nominal variable.

c) {Ice Hockey, Ice skating, Skiing, Cross-country skiing, Sliding, Snowshoe}.

d)

Sport	Frequency	Percent
Ice Hockey	21	0, 35
Ice skating	9	0, 15
Skiing	16	0, 27
Sliding	7	0, 14
Cross-country skiing	2	0, 04
Snowshoe	6	0, 12
Total	60	1

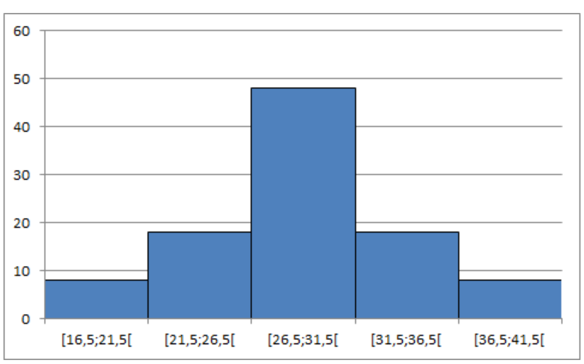


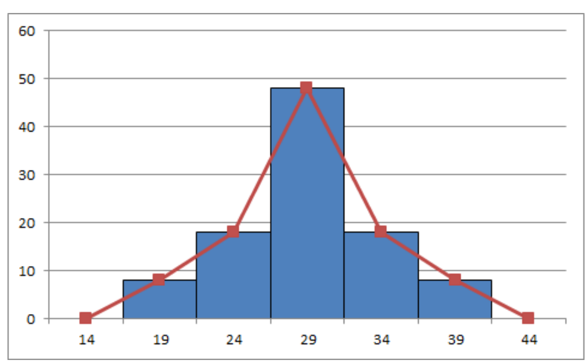
Exercise 11.69.

a)

Classes	Frequency	Percent in %	Cumulative percent in %
$[16,5;21,5[$	4	8	8
$[21,5;26,5[$	9	18	26
$[26,5;31,5[$	24	48	74
$[31,5;36,5[$	9	18	92
$[36,5;41,5[$	4	8	100

b)





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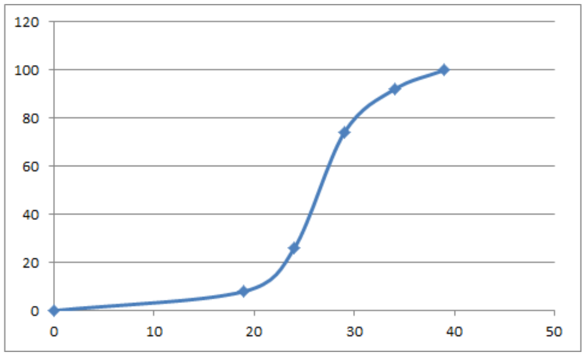
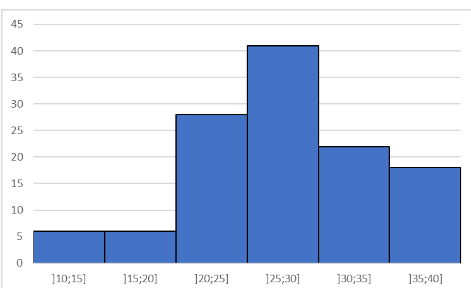


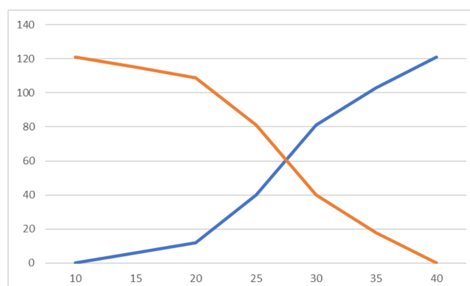
d) 66%.

Exercise 11.70. $\overline{x} = 170,2$, $M_e \in [170;180[$ and $M_o \in [170;180[$.

Exercise 11.71.

a)





b) $M_o \in]25;30]$, $M_e \in]25;30]$ and $\overline{x} = 27,5$ cm.

c) $V \cong 40,702$ cm^2 and $\sigma \cong 6,38$ cm.

d) The current coefficient of variation is $C \cong 23,2\%$, while the coefficient of variation calculated from the previous year's data is given by approximately 21.875%. Therefore, the current population has the highest dispersion.

Bibliography

[1] Hubert Bovet, *Ecole de culture générale Tome 1*, Editions Polymaths, 2018.

[2] Hubert Bovet, *Ecole de culture générale Tome 2*, Editions Polymaths, 2019.

[3] Hubert Bovet, *Algèbre*, Editions Polymaths, 2001.

[4] CRM, *Notions élémentaires*, Editions du Tricorne, 2005.

[5] Jean-Pierre Favre, *Mathématiques pour la maturité professionnelle*, Editions Digilex, 2016.

[6] Peter Frommenwiler, Kurt Studer, *Algèbre et analyse de données*, Cornelsen, 2014.

[7] Olivier Grandjean, *Supports de cours*, CIFOM-ET.

[8] Sylvie Guillod, *Supports de cours*, CIFOM-ET.

[9] Caroline Jacot, *Supports de cours*, CPLN-ET.

[10] Jean-Philippe Javet, *Supports de cours*, Gymnase de Morges.

[11] Juan Perreira, *Supports de cours*, CIFOM-ET.

[12] Stephane Perret, *Supports de cours*, Lycée Cantonal de Porrentruy.

[13] Violaine Sabah, *Supports de cours*, Lycée Jean-Piaget

[14] Karim Saïd, *Statistiques, Filière informatique de gestion*, HEG Arc.

[15] E. W. Swokowski, J. A. Cole, *Algèbre*, Editions LEP, 2006.

[16] Essaïd Zeroual, *Supports de cours*, ESTER.

[17] Maxime Zuber, *Supports de cours*, Gymnase Français de Bienne.

Index